Modelling and entropy satisfying relaxation scheme for the nonconservative bitemperature Euler system with transverse magnetic field

Stéphane Brull *, Xavier Lhébrard†, Bruno Dubroca †

Abstract

The present paper concerns the study of the nonconservative bitemperature Euler system with transverse magnetic field. We firstly introduce an underlying conservative kinetic model coupled to Maxwell equations. The nonconservative bitemperature Euler system with transverse magnetic field is then established from this kinetic model by hydrodynamic limit. Next we present the derivation of a finite volume method to approximate weak solutions. It is obtained by solving a relaxation system of Suliciu type, and is similar to HLLC type solvers. The solver is shown in particular to preserve positivity of density and internal energies. Moreover we use a local minimum entropy principle to prove discrete entropy inequalities, ensuring the robustness of the scheme.

Keywords: BGK models, hydrodynamic limit, relaxation method, nonconservative hyperbolic system, discrete entropy inequalities, discrete entropy minimum principle

Mathematics Subject Classification: 65M08, 35L60, 65M12

1 Introduction

The present paper is devoted to the formal derivation and the approximation of the nonconservative bitemperature Euler system with transverse magnetic field.

This fluid model consists of four conservation equations for mass, momentum (two components), magnetic field, and two nonconservative equations for energies. Physically, this model describes the interaction of a mixture of one species of ions and one species of electrons in thermal nonequilibrium, subjected to a transverse variable magnetic field. The pressure of each species is supposed to satisfy a gamma-law with its own γ constant. Moreover the system owns a dissipative strictly convex entropy, closely related to the classical entropy for the Euler sytem [27].

Solving nonconservative hyperbolic systems is a delicate problem because the definition of weak solutions remains unclear. In order to define nonconservative

*Université de Bordeaux, Institut de Mathématiques de Bordeaux, UMR 5251, F-33405 Talence, France. (Stephane.Brull@u-bordeaux.fr)

†Université de Bordeaux, Centre de Lasers Intenses et Applications, UMR 5107, F- 33400 Talence, France. (Bruno.Dubroca@u-bordeaux.fr), (Xavier.Lhebrard@u-bordeaux.fr)

products, Dal Maso, Lefloch and Murat proposed in [17] a new theory based on the definition of family of paths. In [21], path-conservative schemes are defined by using the concepts developed in [17]. However, it is shown, in [1], that even if the correct path is known, the numerical solution can be far from the expected solution. Let us mention some works dealing with nonconservative Euler type system, where the magnetic fields are neglected. In [5], the authors consider an hyperbolic system having $n-p$ equations in a conservative form, the remaining p equations being nonconservative. In [22], the authors use a Roe solver, and an HLLC solver which neglects the nonconservative part of the system. They validate the approach by comparing their results to theoretical temperatures/pressures curves ([25], [24]). In [16], the authors assume that the electronic entropy is conserved by all weak solutions including shocks and the system is approached by a system of conservation laws.

Despite these difficulties, the nonconservative formulation is physically relevant. Indeed we show that the bitemperature MHD system admits an underlying conservative kinetic model, which consists of a BGK model coupled with Maxwell equation in the quasi-neutral regime. This result generalizes the approach of [2] in the case of a transverse magnetic field. This BGK model possesses different interspecies collision frequencies in order to take into account discrepancies in the masses of the particles. This point has been in particular mentioned in [18], where a conservative formulation is suggested. Hence following this idea we perform an hydrodynamic limit and we formally get the nonconservative MHD system.

In another way, the present bitemperature MHD system is approached numerically. An important issue in multidimensional simulations is to minimize the numerical viscosity by using accurate solvers, in particular on contact discontinuities; while being robust, see for example [19, 28, 6].

If dependency is only one spatial variable x, the nonconservative two species Euler equations with transverse magnetic field are given by:

$$\partial_t\rho + \partial_x\rho u_1 = 0, \quad (1.1)$$
$$\partial_t(\rho u_1) + \partial_x(\rho u_1^2 + p_e + p_i + B_3^2/2) = 0, \quad (1.2)$$
$$\partial_t(\rho u_2) + \partial_x(\rho u_2 u_1) = 0, \quad (1.3)$$
$$\partial_t\overline{\mathcal{E}_e} + \partial_x\left(u_1\left(\overline{\mathcal{E}_e} + p_e + c_e B_3^2/2\right)\right) - u_1\partial_x\left(c_i p_e - c_e p_i\right) = \widetilde{S}_{ei}, \quad (1.4)$$
$$\partial_t\overline{\mathcal{E}_i} + \partial_x\left(u_1\left(\overline{\mathcal{E}_i} + p_i + c_i B_3^2/2\right)\right) + u_1\partial_x\left(c_i p_e - c_e p_i\right) = \widetilde{S}_{ie}, \quad (1.5)$$
$$\partial_t B_3 + \partial_x(u_1 B_3) = 0, \quad (1.6)$$

where $\rho = \rho_e + \rho_e$ is the total density of the plasma, $\vec{u} = (u_1,u_2)$ is the average velocity of the plasma, B_3 is the vertical component of the magnetic field. We denote by T_e, T_i the temperatures of electrons and ions, $\rho_e = n_e m_e$, $\rho_i = n_i m_i$ the densities of the electrons and ions, n_e, n_i the concentrations of the electrons and ions, m_e, m_i, the masses of the electrons and ions particles. The source terms for exchanges between electron and ion species S_{ei} and S_{ie} are defined by

$$\widetilde{S}_{ei} = \widetilde{\nu}_{1,ei}(T_i - T_e) + \widetilde{\nu}_{2,ei}(\partial_x B_3)^2, \quad \widetilde{S}_{ie} = -\widetilde{\nu}_{1,ei}(T_i - T_e) + \widetilde{\nu}_{2,ie}(\partial_x B_3)^2, \quad (1.7)$$

where $\widetilde{\nu}_{1,ei} \geq 0$ is the frequency exchange between temperatures, $\widetilde{\nu}_{2,ei}, \widetilde{\nu}_{2,ie} \geq 0$ are frequencies due to the drift velocy $u_2 \pm \partial_x B_3$. These quantities we will be defined later. The concentrations n_e and n_i are related by the average ionization number $Z = n_e/n_i \geq 1$ which is considered here constant. This implies that the mass fractions $c_\alpha = \rho_\alpha/\rho$, $\alpha = e,i$ are also constant and c_e and c_i write

$$c_e = \frac{Zm_e}{m_i + Zm_e}, \quad c_i = 1 - c_e.$$

The electrons and ions pressures and temperatures are related by

$$p_e = n_e k_B T_e, \quad p_i = n_i k_B T_i,$$

where k_B is the Boltzmann constant. The internal energies are given by

$$\epsilon_e = \frac{k_B T_e}{m_e\,(\gamma_e - 1)}, \quad \epsilon_i = \frac{k_B T_i}{m_i\,(\gamma_i - 1)},$$

where γ_e, γ_i are constant numbers belonging to the interval [1,3]. The quantity

$$\overline{\mathcal{E}_\alpha} = \rho_\alpha \varepsilon_\alpha + \frac{1}{2}\rho_\alpha(u_1^2 + u_2^2) + c_\alpha B_3^2/2$$

is the total energy associated to each species $\alpha = e, i$. This model is closely related to the model derived in [2], the novelty here is to take as an additional variable the vertical component of the magnetic field B_3.

The homogeneous system associated to (1.1) - (1.6) is endowed with an entropy inequality:

$$\partial_t\left(-\rho\left(s_e + s_i\right)\right) + \partial_x\left(-\rho u\left(s_e + s_i\right)\right) \leq 0, \tag{1.8}$$

where s_α, $\alpha = e,i$ is the classical specific entropy

$$s_\alpha(\rho, \varepsilon_\alpha) = \frac{c_\alpha}{m_\alpha(\gamma_\alpha - 1)} \ln\left(\frac{(\gamma_\alpha - 1)\varepsilon_\alpha}{\rho^{(\gamma_\alpha - 1)}}\right) + C. \tag{1.9}$$

Here C is a nonnegative constant.

The homogeneous system associated to (1.1) - (1.6) is an hyperbolic system. The three eigenvalues of the system are u, $u - \sqrt{\left(\gamma_e p_e + \gamma_i p_i + B_3^2\right)/\rho}$ and $u + \sqrt{\left(\gamma_e p_e + \gamma_i p_i + B_3^2\right)/\rho}$ of multiplicity four, one and one, respectively. The eigenvalue u is linearly degenerate and the associated contact discontinuities is called a material contact. The jump relation associated is as follows. Across a material contact, the quantities u_1 and $p_e + p_i + B_3^2/2$ are constant. We note here that unlike the 7-wave full MHD system [11] or the 5-wave shallow water MHD system [12], the transverse magnetic system has a 3-wave structure and does not admit Alfven waves.

A finite volume scheme for this homogeneous quasilinear system is classically built following Godunov's approach, by considering piecewise constant approximation of

$$U = \left(\rho, \rho u_1, \rho u_2, \mathcal{E}_e + c_e B_3^2/2, \mathcal{E}_i + c_i B_3^2/2, B_3\right) \in \mathbb{R}^6 \tag{1.10}$$

and invoking an approximate Riemann solver at the interface between two cells, see for example [20] or [9, Section 2.3]. A difficulty is however that the system is not conservative. In this paper we apply the relaxation approach of [10, 9, 11, 15] to the bitemperature transverse MHD system, in order to get an approximate Riemann solver that is entropy preserving, ensuring robustness, while being exact on isolated material contacts. The relaxation system is of Suliciu type as introduced in [26], and the approximate Riemann solver belongs to the family of HLLC solvers, as in [20, 9, 4, 23, 7, 19, 6].

Let us emphasize that the discrete entropy inequalities are established by arguing a suitable extension of technique introduced in [7], also used in the context of solving the ten moment equations [8]. Put in other words, at the discrepancy with the previous works [10, 9], an entropy extension is obtained implicitly and thus the entropy dissipation is not evaluated in the proposed discrete entropy inequalities.

This work is organized as follows. In the next section we present the kinetic model involved in this paper. Firstly we consider the Vlasov-BGK model from which the MHD system is derived. Starting from an ad-hoc scaling the construction of the MHD system is performed. In section 3 we derive a relaxation scheme. In section 4 we establish the stability of our scheme. Numerical tests are performed in section 5 to illustrate both accuracy and robustness of the proposed scheme in 1D.

2 Kinetic model

Kinetic models are described by the distribution function f_α of each species depending on the time variable $t \in \mathbb{R}_+$, on the positions $x \in \mathbb{R}^3$ and on the velocity $v \in \mathbb{R}^3$. The macroscopic quantities can be obtained by extracting moments on these distribution function w.r.t. the velocity variable. Indeed density, velocity and total energy of the species α can be defined as

$$n_\alpha = \int_{R^3} f_\alpha dv, \quad u_\alpha = \frac{1}{n_\alpha} \int_{R^3} v f_\alpha dv,$$

$$\mathcal{E}_\alpha = \frac{3}{2}\rho_\alpha \frac{k_B}{m_\alpha} T_\alpha + \frac{1}{2}\rho_\alpha u_\alpha^2 = \int_{R^3} m_\alpha \frac{|v|^2}{2} f_\alpha dv,$$

where m_α is the mass particle, $\rho_\alpha = m_\alpha n_\alpha$, and T_α is the temperature of species α.

2.1 Description of the BGK model

In this section, we present the kinetic model and we show the fundamental properties of the BGK model describing the plasma interacting with an electric field $E \in \mathbb{R}^3$ and a magnetic field $B \in \mathbb{R}^3$. The model writes

$$\partial_t f_\alpha + v\nabla_x f_\alpha + \frac{q_\alpha}{m_\alpha}(E + v \wedge B)\nabla_v f_\alpha = \frac{1}{\tau_\alpha}(\mathcal{M}_\alpha(f_\alpha) - f_\alpha) + \frac{1}{\tau_{\alpha\beta}}\left(\overline{\mathcal{M}_\alpha}(f_\alpha, f_\beta) - f_\alpha\right), \quad (2.1)$$

with $\tau_\alpha > 0$, $\tau_{\alpha\beta} > 0$. We denote by $\frac{1}{\tau_\alpha}$ the collision frequency for the interaction between α particles and $\frac{1}{\tau_{\alpha\beta}}$ the collision frequency for the ion/electron interaction. The frequencies τ_{ei} and τ_{ie} being of order of the mass ratios, [18] suggested to take $\tau_{ei} \neq \tau_{ie}$. The quantity q_α is the charge of the species α.

Moreover we define the velocity and the temperature of the mixture by

$$u = \frac{\rho_e u_e + \rho_i u_i}{\rho_e + \rho_i}, \quad T = \frac{\frac{1}{2}\sum_\alpha \rho_\alpha (u_\alpha^2 - u^2) + \frac{3}{2}\sum_\alpha n_\alpha k_B T_\alpha}{\frac{3}{2} n k_B}. \tag{2.2}$$

Let us highlight that one can check that the temperature of the mixture satisfies $T \geq 0$. The two Maxwellian distribution functions $\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$ are defined by

$$\mathcal{M}_\alpha(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T_\alpha / m_\alpha)^{3/2}} \exp(-\frac{|v - u_\alpha|^2}{2 k_B T_\alpha / m_\alpha}), \tag{2.3}$$

and

$$\overline{\mathcal{M}_\alpha}(f_\alpha, f_\beta) = \frac{n_\alpha}{(2\pi k_B T^\# / m_\alpha)^{3/2}} \exp(-\frac{|v - u^\#|^2}{2 k_B T^\# / m_\alpha}). \tag{2.4}$$

Here fictitious velocity and temperature $u^\#$, $T^\#$ are defined by

$$u^\# = \frac{\dfrac{\rho_e}{\tau_{ei}} u_e + \dfrac{\rho_i}{\tau_{ie}} u_i}{\dfrac{\rho_e}{\tau_{ei}} + \dfrac{\rho_i}{\tau_{ie}}}, \quad T^\# = \frac{\frac{1}{2}\sum_\alpha \frac{\rho_\alpha}{\tau_{\alpha\beta}} (u_\alpha^2 - (u^\#)^2) + \frac{3}{2} k_B \sum_\alpha \frac{n_\alpha}{\tau_{\alpha\beta}} T_\alpha}{\frac{3}{2} k_B \left(\frac{n_e}{\tau_{ei}} + \frac{n_i}{\tau_{ie}} \right)}. \tag{2.5}$$

Those fictitious quantities were introduced in [2] where it has been proved that

- $T^\#$ is positive,
- if $u_e = u_i$ and $T_e = T_i$ then $u^\# = u = u_e = u_i$ and $T^\# = T = T_e = T_i$,
- if $\tau_{ei} = \tau_{ie}$ then $u^\# = u$ and $T^\# = T$.

This model is coupled with the Maxwell equations

$$\begin{aligned} \nabla_x \cdot E &= \frac{\bar{\rho}}{\varepsilon_0}, & \partial_t B + \nabla_x \wedge E &= 0, \\ \nabla_x \cdot B &= 0, & \mu_0 \varepsilon_0 \partial_t E - \nabla_x \wedge B &= -\mu_0 j, \end{aligned} \tag{2.6}$$

where $j \in \mathbb{R}^3$ represents the current in the plasma, $\bar{\rho}$ the total charge, c the speed of light and ε_0 is the vacuum permittivity. The quantities j and $\bar{\rho}$ are defined by

$$\bar{\rho} = \int_{\mathbb{R}^3} (q_e f_e + q_i f_i) dv = n_e q_e + n_i q_i, \tag{2.7}$$

$$j = \int_{\mathbb{R}^3} v (q_e f_e + q_i f_i) dv = n_e q_e u_e + n_i q_i u_i. \tag{2.8}$$

2.1.1 Description of the transverse magnetic field case

In this section, we present the particular case where the plasma is interacting with an electric field $E = (E_1, E_2, 0) \in \mathbb{R}^3$ and a magnetic field $B = (0, 0, B_3) \in \mathbb{R}^3$. We use the same notations as in the previous section. The BGK model (2.1) now writes

$$\begin{aligned} \partial_t f_\alpha + v_1 \partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} (E_1 + B_3 v_2) \frac{\partial f_\alpha}{\partial v_1} + \frac{q_\alpha}{m_\alpha} (E_2 - B_3 v_1) \frac{\partial f_\alpha}{\partial v_2} \\ = \frac{1}{\tau_\alpha} (\mathcal{M}_\alpha(f_\alpha) - f_\alpha) + \frac{1}{\tau_{\alpha\beta}} \left(\overline{\mathcal{M}_\alpha}(f_\alpha, f_\beta) - f_\alpha \right). \end{aligned} \tag{2.9}$$

The two Maxwellian distribution functions $\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$ are defined by (2.3), (2.4) and (2.5).
This model is coupled with the Maxwell equations (2.6). In the transverse magnetic field setting (2.6) writes

$$\begin{aligned} \partial_x E_1 &= \frac{\bar\rho}{\varepsilon_0}, & \partial_t E_1 &= -\frac{j_1}{\varepsilon_0}, \\ \partial_t B_3 + \partial_x E_2 &= 0, & \partial_t E_2 - c^2 \partial_x B_3 &= -\frac{j_2}{\varepsilon_0}, \end{aligned} \tag{2.10}$$

where $j = (j_1, j_2)$ represents the current in the plasma, $\bar\rho$ the total charge, c the speed of light and ε_0 is the vacuum permittivity.

2.2 Hydrodynamic limit

2.2.1 Scaling on the one dimensionnal BGK model

We suppose that the distribution function f_α of the species α depends on the time variable $t \in \mathbb{R}_+$, the space variable $x \in \mathbb{R}$ and the velocity variable $v \in \mathbb{R}^3$. In order to use a Chapman-Enskog procedure, we introduce a small parameter ε and the BGK model (2.9) is rescaled as:

$$\begin{aligned} \partial_t f_\alpha + v_1 \partial_x f_\alpha + \frac{q_\alpha}{m_\alpha}(E_1 + B_3 v_2)\frac{\partial f_\alpha}{\partial v_1} + \frac{q_\alpha}{m_\alpha}(E_2 - B_3 v_1)\frac{\partial f_\alpha}{\partial v_2} \\ = \frac{1}{\varepsilon}\left(\mathcal{M}_\alpha(f_\alpha) - f_\alpha\right) + \frac{1}{\tau_{\alpha\beta}}\left(\overline{\mathcal{M}_\alpha}(f_\alpha, f_\beta) - f_\alpha\right). \end{aligned} \tag{2.11}$$

Moreover this model is coupled with Maxwell equations (2.10), which are rescaled as:

$$\begin{aligned} \partial_x E_1 &= \frac{\bar\rho}{\varepsilon^2}, & \partial_t E_1 &= -\frac{j_1}{\varepsilon^2}, \\ \partial_t B_3 + \partial_x E_2 &= 0, & \partial_t E_2 + \frac{1}{\varepsilon^2}\partial_x B_3 &= -\frac{j_2}{\varepsilon^2}. \end{aligned} \tag{2.12}$$

The two Maxwellian distribution functions $\mathcal{M}_\alpha$ and $\overline{\mathcal{M}_\alpha}$ are defined by the following equations:

$$\mathcal{M}_\alpha(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T_\alpha/m_\alpha)^{3/2}} \exp\left(-\frac{(v_1 - u_{1,\alpha})^2 + (v_2 - u_{2,\alpha})^2 + v_3^2}{2 k_B T_\alpha/m_\alpha}\right), \tag{2.13}$$

and

$$\overline{\mathcal{M}_\alpha}(f_\alpha) = \frac{n_\alpha}{(2\pi k_B T^\#/m_\alpha)^{3/2}} \exp\left(-\frac{(v_1 - u_1^\#)^2 + (v_2 - u_2^\#)^2 + v_3^2}{2 k_B T^\#/m_\alpha}\right). \tag{2.14}$$

The two Maxwellian distributions satisfy the constraints

$$\int_{\mathbb{R}^3} (\mathcal{M}_\alpha - f_\alpha) \begin{bmatrix} m_\alpha \\ m_\alpha v \\ m_\alpha \frac{v^2}{2} \end{bmatrix} dv = 0, \quad \int_{\mathbb{R}^3} (\overline{\mathcal{M}_\alpha} - f_\alpha) dv = 0. \tag{2.15}$$

Moreover using the definitions of $u^{\#}$ and $T^{\#}$ given by (2.5) straightforward computations give

$$\int_{\mathbb{R}^3} \left(\frac{1}{\tau_{ei}} (\mathcal{M}_\alpha - f_\alpha) \begin{bmatrix} m_\alpha v \\ m_\alpha \frac{v^2}{2} \end{bmatrix} + \frac{1}{\tau_{ie}} (\overline{\mathcal{M}_\alpha} - f_\alpha) \begin{bmatrix} m_\alpha v \\ m_\alpha \frac{v^2}{2} \end{bmatrix} \right) dv = 0. \tag{2.16}$$

2.2.2 Derivation of Euler equations

In a previous work [2], the nonconservative bitemperature Euler sytem has been established as the hydrodynamic limit of a conservative kinetic system. In particular the authors deal with the case $B_3 = 0$, which implies that the current $j = (j_1, j_2)$ at equilibrium satisfies $j = 0$ and enables to express the electric field as a combination of first order spatial derivatives of the variables ρ, T_e, T_i.
The novelty in our case is that we have $j_1 = 0$ and $j_2 = -\partial_x B_3 \neq 0$. Thus we have a slightly different approach and we proceed in two steps.

- First we perform an hydrodynamic limit of (2.11) -(2.14). The first component E_1 will behave as in [2] and will be expressed as a combination of first order spatial derivatives of the variables ρ, u_2, T_e, T_i, B_3. However, the evolution of E_2 will be given by the second order PDE (2.25).
- Second we use the smallness of the mass ratio m_e/m_i which enables to simplify the equation satisfied by E_2.

After those two steps we recover the bitemperature Euler system with transverse magnetic field (1.1)-(1.6).

Proposition 2.1. *The kinetic conservative system* (2.11) - (2.14) *converges formally to the following system*

$$\partial_t \rho + \partial_x \rho u_1 = 0, \tag{2.17}$$

$$\partial_t(\rho u_1) + \partial_x(\rho u_1^2 + p_e + p_i + B_3^2/2) = 0, \tag{2.18}$$

$$\partial_t(\rho u_2) + \partial_x(\rho u_2 u_1) = 0, \tag{2.19}$$

$$\begin{aligned} \partial_t \left(\mathcal{E}_e + c_e B_3^2/2 \right) + \partial_x \left(u_1 \left(\mathcal{E}_e + p_e \right) + c_e E_2 B_3 \right) - u_1 \partial_x \left(c_i p_e - c_e p_i \right) \\ - \left(E_2 - u_1 B_3 \right) \left(q_e n_e u_2 + (c_e - c_i) \partial_x B_3 \right) = \ S_{ei}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \partial_t \left(\mathcal{E}_i + c_i B_3^2/2 \right) + \partial_x \left(u_1 \left(\mathcal{E}_i + p_i \right) + c_i E_2 B_3 \right) + u_1 \partial_x \left(c_i p_e - c_e p_i \right) \\ + \left(E_2 - u_1 B_3 \right) \left(q_e n_e u_2 + (c_e - c_i) \partial_x B_3 \right) = -S_{ei}, \end{aligned} \tag{2.21}$$

$$\partial_t B_3 + \partial_x E_2 = 0, \tag{2.22}$$

where

$$\mathcal{E}_\alpha = \rho_\alpha \varepsilon_\alpha + \frac{1}{2} \rho_\alpha (u_1^2 + u_{2,\alpha}^2), \quad \alpha = e,i \tag{2.23}$$

with $u_{2,\alpha}$ *defined by* (2.37)*. Moreover the first component of electric field* E_1 *is explicitly given by the Ohm's law*

$$c_i \partial_x p_e - c_e \partial_x p_i + (c_i - c_e) \partial_x (B_3^2/2) = q_e n_e \left(E_1 + u_2 B_3 \right), \tag{2.24}$$

and the second component of electric field E_2 *satisfies the following equation:*

$$\partial_t(\partial_x B_3) + \partial_x(u_1 \partial_x B_3) + (q_e n_e)^2 \frac{\rho}{\rho_e \rho_i} (E_2 - B_3 u_1) = -\frac{\partial_x B_3}{\tau_{ie} c_e + \tau_{ei} c_i}. \tag{2.25}$$

In addition the source term S_{ei} is defined by

$$S_{ei} = \nu_{1,ei}(T_i - T_e) + \nu_{2,ei}(\partial_x B_3)^2 + \nu_{3,ei} u_2 \partial_x B_3, \tag{2.26}$$

(1.7) *with the coefficients $\nu_{1,ei}$, $\nu_{2,ei}$, $\nu_{3,ei}$ given by*

$$\nu_{1,ei} = \frac{3}{2}\frac{k_B n_e n_i}{\tau_{ie} n_e + \tau_{ei} n_i}, \tag{2.27}$$

$$\nu_{2,ei} = c_e c_i \rho \frac{(c_e - c_i)\left(n_i c_i \tau_{ei}^2 + n_e c_e \tau_{ie}^2\right) - 2c_e c_i (n_i - n_e)\tau_{ei}\tau_{ie}}{(\tau_{ie} n_e + \tau_{ei} n_i)\,(q_e n_e)^2\,(\tau_{ie} c_e + \tau_{ei} c_i)^2}, \tag{2.28}$$

and

$$\nu_{3,ei} = \frac{c_e c_i \rho}{(q_e n_e)\,(c_e \tau_{ie} + c_i \tau_{ei})}. \tag{2.29}$$

Note that with Proposition 2.1, we do not recover yet (1.1) - (1.6). Using the following approximation we are able to simplify the previous system and recover the bitemperature Euler system with transverse magnetic field.

Proposition 2.2. *We consider the previous system* (2.17)-(2.29). *Using the approximation of a small mass ratio between electron and ion species $m_e/m_i = \epsilon << 1$, the equation* (2.25) *formally converges to*

$$E_2 = B_3 u_1. \tag{2.30}$$

Thus in this setting the conservative kinetic system (2.11) - (2.14) *formally converges to the nonconservative bitemperature MHD system* (1.1) - (1.6). *The electric field $E = (E_1, E_2)$ is given by the Ohm's laws* (2.24) *and* (2.30). *The exchange coefficient $\widetilde{\nu}_{1,ei} = \nu_{1,ei}$ is defined by* (2.27). *The magnetic coefficients $\widetilde{\nu}_{2,ei}$, $\widetilde{\nu}_{2,ie}$ are defined by*

$$\widetilde{\nu}_{2,ei} = c_e c_i \rho \frac{(\tau_{ie} n_e + \tau_{ei} n_i)\,(\tau_{ie} c_e + \tau_{ei} c_i) + (c_i n_e - c_e n_i)\,\tau_{ei}\tau_{ie}}{2(q_e n_e)^2\,(\tau_{ie} n_e + \tau_{ei} n_i)\,(\tau_{ie} c_e + \tau_{ei} c_i)^2}, \tag{2.31}$$

$$\widetilde{\nu}_{2,ie} = c_e c_i \rho \frac{(\tau_{ie} n_e + \tau_{ei} n_i)\,(\tau_{ie} c_e + \tau_{ei} c_i) - (c_i n_e - c_e n_i)\,\tau_{ei}\tau_{ie}}{2(q_e n_e)^2\,(\tau_{ie} n_e + \tau_{ei} n_i)\,(\tau_{ie} c_e + \tau_{ei} c_i)^2}. \tag{2.32}$$

Remark 2.3. *In the case $B_3 = 0$ we recover all the results of [2]. Concerning the Ohms law, compared to [2], in* (2.24) *we have additional terms* $(c_e - c_i)\partial_x(B_3^2/2) + q_e n_e u_2 B_3$.

Proof of Proposition 2.1. For this proof, for any g belonging in $L^1_2 = \{g \in L^1/(1+v^2)f \in L^1\}$, we use the notation

$$\langle g \rangle = \int_{\mathbb{R}^3} g dv.$$

the equilibrium states of (2.11) write

$$f_\alpha = \mathcal{M}_\alpha, \quad \alpha = e,i, \quad \bar{\rho} = 0, \quad j_1 = 0,$$
$$\partial_t B_3 + \partial_x E_2 = 0, \quad j_2 = -\partial_x B_3.$$

This system implies that

$$n_e q_e + n_i q_i = 0, \quad n_e q_e u_{1,e} + n_i q_i u_{1,i} = 0,$$
$$\partial_t B_3 + \partial_x E_2 = 0, \quad n_e q_e u_{2,e} + n_i q_i u_{2,i} = -\partial_x B_3,$$

which is equivalent to

$$n_e q_e + n_i q_i = 0, \tag{2.33}$$
$$u_{1,e} = u_{1,i} = u, \tag{2.34}$$
$$n_e q_e \left(u_{2,e} - u_{2,i}\right) = -\partial_x B_3, \tag{2.35}$$
$$\partial_t B_3 + \partial_x E_2 = 0, \tag{2.36}$$

with u defined by (2.2). Moreover combining (2.2) and (2.35) we compute

$$\begin{cases} u_{2,e} = u_2 - \dfrac{c_i}{q_e n_e}\partial_x B_3, \\ u_{2,i} = u_2 - \dfrac{c_e}{q_i n_i}\partial_x B_3. \end{cases} \tag{2.37}$$

Next we establish the hydrodynamic limit associated to the previous equilibrium. Hence f_α writes

$$f_\alpha = \mathcal{M}_\alpha + \varepsilon g_\alpha, \tag{2.38}$$

with

$$\int_{\mathbb{R}^3} m_\alpha \begin{pmatrix} 1 \\ |v|^2 \end{pmatrix} g_\alpha = 0, \quad \int_{\mathbb{R}^3} m_\alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} g_\alpha = 0. \tag{2.39}$$

Plugging (2.38) into (2.11), we compute g_α as follows

$$\begin{aligned} \partial_t \mathcal{M}_\alpha + v_1 \partial_x \mathcal{M}_\alpha + \frac{q_\alpha}{m_\alpha}\left(E_1 + B_3 v_2\right)\partial_{v_1}\mathcal{M}_\alpha + \frac{q_\alpha}{m_\alpha}\left(E_2 - B_3 v_1\right)\partial_{v_2}\mathcal{M}_\alpha \\ = -g_\alpha + \frac{1}{\tau_{\alpha\beta}}\left(\overline{\mathcal{M}_\alpha} - \mathcal{M}_\alpha\right) + O(\varepsilon). \end{aligned} \tag{2.40}$$

From here we will take the moments of the previous equation. The procedure we use here is very similar to the one detailed in [2]. So we skip some details of computation.

Our goal is to obtain the system (2.17)- (2.22) with E_1, E_2 defined by (2.24), (2.25), respectively.

a) First we proceed similarly to [2] we multiply by m_α and we integrate w.r.t. v (2.40) and we obtain

$$\partial_t \rho_\alpha + \partial_x(\rho_\alpha u_1) = 0 \tag{2.41}$$

and by summing for electrons and ions we get (2.17).

b) Then we multiply by $m_\alpha v_1$ we integrate w.r.t. v (2.40) and we obtain

$$\partial_t(\rho_\alpha u_1) + \partial_x(\rho_\alpha u_1^2 + p_\alpha) - q_\alpha n_\alpha \left(E_1 + B_3 u_{2,\alpha}\right) = 0. \tag{2.42}$$

After integration the RHS vanishes because using (2.34) and the definition of $u^\#$ in (2.5) , we get that $u_{1,e} = u_{1,i} = u_1^\# = u_1$. Thus we obtain that

$$\frac{1}{\tau_{\alpha\beta}}\left\langle m_\alpha v_1 \left(\overline{\mathcal{M}_\alpha} - \mathcal{M}_\alpha\right)\right\rangle = 0.$$

By summing for electrons and ions (2.42) and using (2.33), (2.34) we get (2.18).

Next since $u_{1,e} = u_{1,i}$, we get the generalized Ohm's law for E_1 by following the strategy used in [2].

First using (2.41) we get the following nonconservative form of (2.42):

$$\partial_t u_1 + u_1 \partial_x u_1 + \frac{1}{\rho_\alpha} \partial_x p_\alpha - \frac{q_\alpha n_\alpha}{\rho_\alpha} (E_1 + B_3 u_{2,\alpha}) = 0.$$

From here we proceed as in [2] with the additional term $\frac{q_\alpha n_\alpha}{\rho_\alpha} B_3 u_{2,\alpha}$. In the end we get (2.24).

c) Then we multiply by $m_\alpha v_2$ and we integrate w.r.t. v:

$$\partial_t(\rho_\alpha u_{2,\alpha}) + \partial_x(\rho_\alpha u_{2,\alpha} u_1) - q_\alpha n_\alpha (E_2 - B_3 u_1) = \frac{\rho_\alpha}{\tau_{\alpha\beta}} \left(u_2^{\#} - u_{2,\alpha} \right). \quad (2.43)$$

Using (2.33) and the definition of $u^{\#}$ in (2.2), we sum for electrons and ions (2.43) and we get (2.19).

Next we compute a generalized Ohm's law for E_2. First we use (2.35) and the definition of $u^{\#}$ in (2.5) in order to obtain that

$$\frac{1}{\tau_{\alpha\beta}} \left(u_2^{\#} - u_{2,\alpha} \right) = \frac{c_\beta}{c_\alpha \tau_{\beta\alpha} + c_\beta \tau_{\alpha\beta}} \left(\frac{\partial_x B_3}{q_\alpha n_\alpha} \right). \quad (2.44)$$

Then we multiply (2.43) by q_α / m_α and we use (2.44) to get

$$\partial_t(q_\alpha n_\alpha u_{2,\alpha}) + \partial_x(q_\alpha n_\alpha u_{2,\alpha} u_1) - \frac{(q_\alpha n_\alpha)^2}{\rho_\alpha} (E_2 - B_3 u_1) = \frac{c_\beta \partial_x B_3}{c_\alpha \tau_{\beta\alpha} + c_\beta \tau_{\alpha\beta}}. \quad (2.45)$$

Finally using (2.33) and $q_e n_e u_{2,e} + q_i n_i u_{2,i} = -\partial_x B_3$, we sum for electrons and ions (2.45) and we obtain (2.25).

d) Then we multiply by $m_\alpha \frac{|v|^2}{2}$ and we integrate w.r.t. v:

$$\begin{aligned} \partial_t \mathcal{E}_\alpha + \partial_x(u_1(\mathcal{E}_\alpha + p_\alpha)) - q_\alpha n_\alpha (E_1 + B_3 u_{2,\alpha}) u_1 & \\ - q_\alpha n_\alpha (E_2 - B_3 u_1) u_{2,\alpha} &= S, \end{aligned} \quad (2.46)$$

whith $\mathcal{E}_\alpha$ defined by (2.23) and

$$S = \frac{1}{\tau_{\alpha\beta}} \left\langle m_\alpha \frac{|v|^2}{2} \left(\overline{\mathcal{M}_\alpha} - \mathcal{M}_\alpha \right) \right\rangle. \quad (2.47)$$

Using (2.24) and (2.37) we get

$$\begin{aligned} \partial_t \mathcal{E}_\alpha + \partial_x(u_1(\mathcal{E}_\alpha + p_\alpha)) + u_1 (c_\alpha \partial_x p_\beta - c_\beta \partial_x p_\alpha) + u_1 c_\alpha B_3 \partial_x B_3 & \\ - (E_2 - B_3 u_1)(q_\alpha n_\alpha u_2 - c_\beta \partial_x B_3) &= S. \end{aligned} \quad (2.48)$$

Moreover, we multiply (2.36) by $c_\alpha B_3$ and we get

$$\partial_t(c_\alpha B_3^2/2) + \partial_x(c_\alpha B_3 E_2) - c_\alpha E_2 \partial_x B_3 = 0. \quad (2.49)$$

Adding (2.48) and (2.49) we get, for $\alpha \neq \beta$

$$\partial_t \left(\mathcal{E}_\alpha + c_\alpha B_3^2/2\right) + \partial_x(u_1(\mathcal{E}_\alpha + p_\alpha) + c_\alpha E_2 B_3) + u_1\left(c_\alpha \partial_x p_\beta - c_\beta \partial_x p_\alpha\right) \\ - (E_2 - B_3 u_1)\left(q_\alpha n_\alpha u_2 + (c_\alpha - c_\beta)\partial_x B_3\right) = S. \quad (2.50)$$

At this point we deal with S defined by (2.47). Using (2.13) and (2.14) we write

$$S = \frac{3}{2}\frac{k_B n_\alpha}{\tau_{\alpha\beta}}\left(T^{\#} - T_\alpha\right) + \frac{1}{2}\frac{\rho_\alpha}{\tau_{\alpha\beta}}\left((u_2^{\#})^2 - u_{2,\alpha}^2\right). \quad (2.51)$$

Then from the definition of $T^{\#}$ in (2.5) on can write

$$T^{\#} - T_\alpha = \frac{\tau_{\alpha\beta} n_\beta}{\tau_{\beta\alpha} n_\alpha + \tau_{\alpha\beta} n_\beta}\left(T_\beta - T_\alpha\right) \\ + \frac{\tau_{\alpha\beta}\tau_{\beta\alpha}}{3k_B\left(\tau_{\beta\alpha} n_\alpha + \tau_{\alpha\beta} n_\beta\right)}\sum_\alpha \frac{\rho_\alpha}{\tau_{\alpha\beta}}\left(u_{2,\alpha}^2 - (u_2^{\#})^2\right),$$

where $(\alpha, \beta) \in \{e, i\}$, $\alpha \neq \beta$. Next we use the last equation in (2.51), and we get

$$S = \frac{3}{2}\frac{k_B n_\alpha n_\beta}{\tau_{\beta\alpha} n_\alpha + \tau_{\alpha\beta} n_\beta}\left(T_\beta - T_\alpha\right) \\ + \frac{1}{2}\frac{n_\alpha n_\beta}{\tau_{\beta\alpha} n_\alpha + \tau_{\alpha\beta} n_\beta}\left(m_\alpha\left((u_2^{\#})^2 - u_{2,\alpha}^2\right) - m_\beta\left((u_2^{\#})^2 - u_{2,\beta}^2\right)\right). \quad (2.52)$$

Now we deal with $(u_2^{\#})^2 - u_{2,\alpha}^2$. First using (2.37) in the definition of $u^{\#}$ in (2.5) we write

$$u_2^{\#} = u_2 + \frac{c_\alpha c_\beta \partial_x B_3\left(\tau_{\alpha\beta} - \tau_{\beta\alpha}\right)}{(q_\alpha n_\alpha)\left(c_\alpha \tau_{\beta\alpha} + c_\beta \tau_{\alpha\beta}\right)}.$$

Moreover using (2.37) we can compute $u_2^{\#} - u_{2,\alpha}$ and $u_2^{\#} + u_{2,\alpha}$ and after some simple computations we get that

$$m_\alpha\left((u_2^{\#})^2 - u_{2,\alpha}^2\right) = \frac{2 m_\alpha c_\beta \tau_{\alpha\beta}}{q_\alpha n_\alpha\left(c_\alpha \tau_{\beta\alpha} + c_\beta \tau_{\alpha\beta}\right)} u_2 \partial_x B_3 \\ + \frac{m_\alpha c_\beta^2 \tau_{\alpha\beta}\left((c_\alpha - c_\beta)\tau_{\alpha\beta} - 2 c_\alpha \tau_{\beta\alpha}\right)}{(q_\alpha n_\alpha)^2\left(c_\alpha \tau_{\beta\alpha} + c_\beta \tau_{\alpha\beta}\right)^2}\left(\partial_x B_3\right)^2.$$

At this point we sum last equality for electron and ion species and we get

$$\frac{1}{2}\frac{n_e n_i}{\tau_{ie} n_e + \tau_{ei} n_i}\left(m_e\left((u_2^{\#})^2 - u_{2,e}^2\right) - m_i\left((u_2^{\#})^2 - u_{2,i}^2\right)\right) \\ = \nu_{2,ei}(u_2 \partial_x B_3) + \nu_{3,ei}(\partial_x B_3)^2, \quad (2.53)$$

with $\nu_{2,ei}$, $\nu_{3,ei}$ defined by (2.28), (2.29).
Plugging (2.53) into (2.52) we get the RHS of (2.20) with $\nu_{1,ei}$, $\nu_{2,ei}$ and $\nu_{3,ei}$ defined by (2.27), (2.28) and (2.29), which concludes the proof.

□

Proof of Proposition 2.2. We use Proposition 2.1 and we deal with the system (2.17) - (2.25). In addition we use the small mass ratio assumption $\varepsilon = m_e/m_i$. We also use the constant ionization number $Z = n_e/n_i$ and we get

$$\frac{\rho}{\rho_e \rho_i} = \frac{n_e m_e + n_i m_i}{n_e m_e n_i m_i} = \frac{(\varepsilon Z + 1)}{(m_i n_i)\varepsilon Z} = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{2.54}$$

Retaining terms of order 0 in (2.25), it leads to (2.30). Moreover using (2.30) in (2.20) - (2.22), we get

$$\begin{aligned} \partial_t \left(\mathcal{E}_e + c_e B_3^2/2\right) + \partial_x \left(u_1 \left(\mathcal{E}_e + c_e B_3^2/2 + p_e + c_e B_3^2/2\right)\right) & \\ - u_1 \partial_x \left(c_i p_e - c_e p_i\right) = \ S_{ei}, & \qquad (2.55) \\ \partial_t \left(\mathcal{E}_i + c_i B_3^2/2\right) + \partial_x \left(u_1 \left(\mathcal{E}_i + c_i B_3^2/2 + p_i + c_i B_3^2/2\right)\right) & \\ + u_1 \partial_x \left(c_i p_e - c_e p_i\right) = -S_{ei}, & \qquad (2.56) \\ \partial_t B_3 + \partial_x \left(u_1 B_3\right) = 0, & \qquad (2.57) \end{aligned}$$

with S_{ei} defined by (2.26)-(2.29). To complete the proof we have to recover (1.4)- (1.5) with $\widetilde{\nu}_{1,ei} = \nu_{1,ei}$, $\nu_{1,ei}$ defined by (2.27), and $\widetilde{\nu}_{2,ei}$, $\widetilde{\nu}_{2,ie}$ defined by (2.31), (2.32). In the following we focus on the electron energy equation (1.4), the ion energy equation (1.5) is obtained in a similar way. First we notice the following equality:

$$\overline{\mathcal{E}_e} = \mathcal{E}_e + c_e B_3^2/2 + \rho_e \frac{u_2^2}{2} - \rho_e \frac{u_{2,e}^2}{2}. \tag{2.58}$$

From (1.4) we have a PDE on $\mathcal{E}_e + c_e B_3^2/2$, thus in order to use the previous relation we need PDEs on $\rho_e \frac{u_2^2}{2}$ and $\rho_e \frac{u_{2,e}^2}{2}$. Thus on the one hand, we use (2.30) and (2.43) in order to get

$$\partial_t(\rho_e u_{2,e}) + \partial_x(\rho_e u_{2,e} u_1) = \nu_{3,ei} \partial_x B_3, \tag{2.59}$$

with $\nu_{3,ei}$ defined by (2.29). Next we multiply by $u_{2,e}$ (2.59) and we use (2.17) in order to get that

$$\partial_t \left(\rho_e \frac{u_{2,e}^2}{2}\right) + \partial_x \left(\rho_e \frac{u_{2,e}^2}{2} u_1\right) = -\frac{c_i}{q_e n_e} \nu_{3,ei} (\partial_x B_3)^2 + \nu_{3,ei} u_2 \partial_x B_3. \tag{2.60}$$

On the other hand, we combine (2.17), (2.19) and we obtain

$$\partial_t \left(\rho_e \frac{u_2^2}{2}\right) + \partial_x \left(\rho_e \frac{u_2^2}{2} u_1\right) = 0, \tag{2.61}$$

Next following (2.58), we add (2.55), (2.61) and we substract (2.60), it leads to (1.4). Moreover we obtain the following formulas for $\widetilde{\nu}_{1,ei}$, $\widetilde{\nu}_{2,ei}$, $\widetilde{\nu}_{2,ie}$:

$$\widetilde{\nu}_{1,ei} = \nu_{1,ei}, \quad \widetilde{\nu}_{2,ei} = \nu_{2,ei} + \frac{c_i}{q_e n_e} \nu_{3,ei}, \quad \widetilde{\nu}_{2,ie} = -\nu_{2,ei} + \frac{c_e}{q_e n_e} \nu_{3,ei}.$$

Finally using (2.28), (2.29) one can recover (2.31). □

3 Numerical approximation

The numerical approximation we use has two steps:

- **First step.** We use an approximate Riemann solver for the homogeneous system. Let $\overline{\mathcal{U}}^{n+1/2}$ be the obtained solution.
- **Second step.** We take the temperatures interaction into account implicitly: the approximate solution of system at time t^{n+1} is defined by

$$\rho^{n+1} = \overline{\rho}^{n+1/2},\ u_1^{n+1} = \overline{u_1}^{n+1/2},\ u_2^{n+1} = \overline{u_2}^{n+1/2},\ B_3^{n+1} = \bar{B}_3^{n+1/2}$$

and

$$\begin{cases} \mathcal{E}_e^{n+1} = \overline{\mathcal{E}_e}^{n+1/2} + \Delta t\, S_{ei}^{n+1}, \\ \mathcal{E}_i^{n+1} = \overline{\mathcal{E}_i}^{n+1/2} - \Delta t\, S_{ei}^{n+1}, \end{cases}$$

where

$$S_{ei}^{n+1} = \widetilde{\nu}_{1,ei}(T_i^{n+1} - T_e^{n+1}) + \widetilde{\nu}_{2,ei}\left((\partial_x B_3)^{n+1}\right)^2.$$

This system is linear and owns an explicit solution.

From here we will focus on the first step of the numerical approximation. We will explain how to derive an efficient approximate Riemann solvers for the homogeneous part of the system (1.1) - (1.6).

In order to get those approximate Riemann solvers, we use a standard relaxation approach, introduced in [9] for the gas dynamic equations. This approach has been developed in [11] for the MHD equations, in [3] for shallow elastic fluids, in [12] for shallow water MHD equations, in [2] for the bitemperature Euler system. An abstract general description can be found in [15], and related works are [13, 14]. This technique enables to naturally handle the entropy inequality (1.8), and also preserves the positivity of density and internal energies.

3.1 Relaxation approach

3.1.1 Approximate Riemann solver

We introduce new variables π_e,π_i, the relaxed pressures, and a intended to parametrize the speed. The form of the relaxation system is as follows,

$$\partial_t\rho + \partial_x\rho u_1 = 0, \quad (3.1)$$

$$\partial_t(\rho u_1) + \partial_x(\rho u_1^2 + \pi_e + \pi_i + B_3^2/2) = 0, \quad (3.2)$$

$$\partial_t(\rho u_2) + \partial_x(\rho u_2 u_1) = 0, \quad (3.3)$$

$$\partial_t\overline{\mathcal{E}_e} + \partial_x\left(u_1\left(\overline{\mathcal{E}_e} + \pi_e + c_e B_3^2/2\right)\right) - u_1\partial_x\left(c_i\pi_e - c_e\pi_i\right) = 0, \quad (3.4)$$

$$\partial_t\overline{\mathcal{E}_i} + \partial_x\left(u_1\left(\overline{\mathcal{E}_i} + \pi_i + c_i B_3^2/2\right)\right) + u_1\partial_x\left(c_i\pi_e - c_e\pi_i\right) = 0, \quad (3.5)$$

$$\partial_t B_3 + \partial_x(u_1 B_3) = 0, \quad (3.6)$$

$$\partial_t(\rho\pi_e) + \partial_x(\rho u_1\pi_e) + c_e\left(a^2 - \rho B_3^2\right)\partial_x u_1 = 0, \quad (3.7)$$

$$\partial_t(\rho\pi_i) + \partial_x(\rho u_1\pi_i) + c_i\left(a^2 - \rho B_3^2\right)\partial_x u_1 = 0, \quad (3.8)$$

$$\partial_t(\rho a) + \partial_x(\rho a u_1) = 0. \quad (3.9)$$

The approximate Riemann solver can be defined as follows, starting from left and right values U_l, U_r at an interface.

- Solve the Riemann problem of the system (3.1)-(3.9) with initial data obtained by completing U_l, U_r by the equilibrium relations

$$\begin{array}{rcccl} \pi_{e,L} & = & p_{e,L} & \equiv & (\gamma_e - 1)\rho_L \varepsilon_{e,L}, \\ \pi_{i,L} & = & p_{i,L} & \equiv & (\gamma_i - 1)\rho_L \varepsilon_{i,L}, \\ \pi_{e,R} & = & p_{e,R} & \equiv & (\gamma_e - 1)\rho_R \varepsilon_{e,R}, \\ \pi_{i,R} & = & p_{i,R} & \equiv & (\gamma_i - 1)\rho_R \varepsilon_{i,R}, \end{array} \tag{3.10}$$

and with suitable positive values of a_l, a_r that will be discussed further on, essentially in Section 3.2.1.

- Retain in the solution only the variables ρ, ρu_1, ρu_2, $\overline{\mathcal{E}_e}$, $\overline{\mathcal{E}_i}$, B_3. The result is a vector called $R(x/t, U_l, U_r)$.

Intuitively, the solver is consistent because of the equations (3.1)-(3.6), that are consistent with the equations (1.1)-(1.6). The specific values used for a do not play any role in this consistency.

The accuracy of the solver on isolated contacts is described by the following lemma.

Lemma 3.1. *The approximate Riemann solver $R(x/t, U_l, U_r)$ solves exactly the material contact discontinuities.*

Proof. Material contacts are solutions to the bitemperature MHD system (1.1) - (1.6) with u_1, $p_e + p_i + B_3^2/2$ constant. These solutions are obviously solutions to the relaxation system (3.1) - (3.8) with $\pi_e = p_e$, $\pi_i = p_i$. Thus for these data, R coincides with the exact solver, which concludes the proof. □

3.1.2 Godunov scheme

Following the Godunov approach, the numerical scheme can be defined by the approximate Riemann solver as follows. We consider a mesh of cells $(x_{i-1/2}, x_{i+1/2})$, $i \in \mathbb{Z}$, of length $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, discrete times t_n with $t_{n+1} - t_n = \Delta t$, and cell values U_i^n approximating the average of U over the cell i at time t_n. We can then define an approximate solution $U_{appr}(t,x)$ for $t_n \leq t < t_{n+1}$ and $x \in \mathbb{R}$ by

$$U_{appr}(t,x) = R(\frac{x - x_{i+1/2}}{t - t_n}, U_i^n, U_{i+1}^n) \text{ for } x_i < x < x_{i+1}, \tag{3.11}$$

where $x_i = (x_{i-1/2} + x_{i+1/2})/2$. This definition is coherent under a half CFL condition, formulated as

$$\begin{aligned} x/t < -\frac{\Delta x_i}{2\Delta t} &\Rightarrow R(x/t, U_i, U_{i+1}) = U_i, \\ x/t > \frac{\Delta x_{i+1}}{2\Delta t} &\Rightarrow R(x/t, U_i, U_{i+1}) = U_{i+1}. \end{aligned} \tag{3.12}$$

The new values at time t_{n+1} are defined by

$$U_i^{n+1} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} U_{appr}(t_{n+1} - 0, x) dx.$$

Notice that it is only in this averaging procedure that the choice of the particular pseudo-conservative variable U as (1.10) is involved. We can follow the computations of [9, Section 2.3], the only difference being that the system is not conservative. We obtain the update formula

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x_i}(\mathcal{F}_l(U_i^n, U_{i+1}^n) - \mathcal{F}_r(U_{i-1}^n, U_i^n)), \tag{3.13}$$

where

$$\begin{aligned}\mathcal{F}_l(U_l,U_r) &= F(U_l) - \int_{-\infty}^{0} (R(\xi, U_l, U_r) - U_l) d\,\xi, \\ \mathcal{F}_r(U_l,U_r) &= F(U_r) + \int_{0}^{\infty} (R(\xi, U_l, U_r) - U_r)\, d\xi.\end{aligned} \tag{3.14}$$

The variable ξ stands for x/t, and the pseudo-conservative flux is chosen as

$$\begin{aligned}F(U) \equiv \big(\rho u_1, \quad \rho u_1^2 + p_e + p_i + B_3^2/2, \quad \rho u_1 u_2, \\ u_1\left(\overline{\mathcal{E}_e} + p_e + c_e B_3^2/2\right), \quad u_1\left(\overline{\mathcal{E}_e} + p_e + c_e B_3^2/2\right), \quad u_1 B_3\big).\end{aligned} \tag{3.15}$$

In (3.15), the fourth and fifth components could be chosen differently since the two energy equations in our system are not conservative. We can remark that the choice of F has no influence on the update formula (3.13).

3.1.3 Subcharacteristic condition

We are interested on necessary stability conditions for smooth solutions. In order to adress such an issue, we consider (3.7), (3.8) in which we add right-hand-sides:

$$\partial_t(\rho\pi_e) + \partial_x(\rho u_1 \pi_e) + c_e\left(a^2 - \rho B_3^2\right)\partial_x u_1 = \frac{\rho}{\tau}\left(p_e - \pi_e\right), \tag{3.16}$$

$$\partial_t(\rho\pi_i) + \partial_x(\rho u_1 \pi_i) + c_i\left(a^2 - \rho B_3^2\right)\partial_x u_1 = \frac{\rho}{\tau}\left(p_i - \pi_i\right). \tag{3.17}$$

On the other hand one can check with straightforward computations that the smooth solutions to (1.1) - (1.6) verify

$$\partial_t(\rho p_e) + \partial_x(\rho p_e u_1) + \gamma_e \rho p_e \partial_x u_1 = 0, \tag{3.18}$$

$$\partial_t(\rho p_i) + \partial_x(\rho p_i u_1) + \gamma_i \rho p_i \partial_x u_1 = 0. \tag{3.19}$$

Next when the relaxation parameter τ tends to zero one has

$$\pi_\alpha = p_\alpha + \delta\tau + O(\tau^2), \quad \alpha = e,i. \tag{3.20}$$

Pluggin (3.20) into (3.16) or (3.17) then substracting (3.18)-(3.19) we get that :

$$\delta = c_\alpha\left(\rho B_3^2 + \gamma_\alpha \rho p_\alpha - a^2\right)\partial_x u_1 + O(\tau).$$

Hence the stability condition needed by the parameter a is

$$a^2 > \rho B_3^2 + \gamma_e \rho p_e + \gamma_i \rho p_i. \tag{3.21}$$

3.1.4 Intermediate states

By using the variable $V = (\rho, u_1, u_2, B_3, \varepsilon_e, \varepsilon_i, \pi_e, \pi_i, a)$, one can easily compute the eigenvalues of the system (3.1) - (3.9). They read as $\{u - a/\rho, u, u + a/\rho\}$, where u is an eigenvalue of order 7. All the fields are linearly degenerated. As a consequence, Rankine-Hugoniot conditions are well-defined (the weak Riemann invariants do not jump through the associated discontinuity), and are equivalent to any conservative formulation.
In the solution to the Riemann problem, the speeds corresponding to the previous eigenvalues will be denoted by

$$\Sigma_1 < \Sigma_2 < \Sigma_3. \tag{3.22}$$

Thus we get a 3-wave solver with two intermediate states. The variables take the values "L" for $x/t < \Sigma_1$, "L*" for $\Sigma_1 < x/t < \Sigma_2$, "R*" for $\Sigma_2 < x/t < \Sigma_3$, "R" for $\Sigma_3 < x/t$, see Figure 1. There are 7 strong Riemann invariants for the

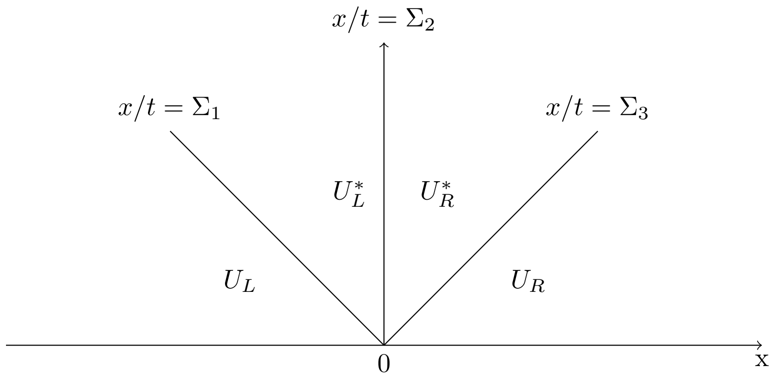

Figure 1: Intermediate states in the Riemann solution

central wave (i.e. quantities that lie in the kernel of $\partial_t + u_1\partial_x$), which are

$$u_2, \quad a, \quad \frac{B_3}{\rho}, \quad w_{1,e}, \quad w_{1,i}, \quad w_{2,e}, \quad w_{2,i}, \tag{3.23}$$

with $w_{1,e}, w_{1,i}, w_{2,e}, w_{2,i}$ defined by

$$w_{1,\alpha} = \pi_\alpha + c_\alpha B_3^2/2 + \frac{a_1^2 c_\alpha}{\rho}, \quad w_{2,\alpha} = \varepsilon_\alpha + \frac{B_3^2}{2\rho} - \frac{\left(\pi_\alpha + c_\alpha B_3^2/2\right)^2}{2(c_\alpha\, a_1)^2}.$$

These quantities are thus weak Riemann invariants for the other waves. Two weak Riemann invariants for the central wave are

$$u_1, \quad \pi, \tag{3.24}$$

with $\pi = \pi_e + \pi_i + B_3^2/2$. We shall denote u_1^* the common value of u_1 on the left and on the right of the central wave.
Computations using the Riemann invariants (3.23), (3.24) give the following

intermediate states:

$$\begin{cases} \dfrac{1}{\rho_L^*} = \dfrac{1}{\rho_L} + \dfrac{a_R(u_{1,R}-u_{1,L})+\pi_L-\pi_R}{a_L(a_L+a_R)}, \\ \dfrac{1}{\rho_R^*} = \dfrac{1}{\rho_R} + \dfrac{a_L(u_{1,R}-u_{1,L})+\pi_R-\pi_L}{a_R(a_L+a_R)}, \end{cases}$$

$$u_1^* = \frac{a_R u_{1,R} + a_L u_{1,L} + \pi_L - \pi_R}{a_L + a_R}, \tag{3.25}$$

$$B_{3,L}^* = \frac{\rho_L^*}{\rho_L} B_{3,L}, \quad B_{3,R}^* = \frac{\rho_R^*}{\rho_R} B_{3,R},$$

for $\alpha = e,i$ we have

$$\begin{cases} \pi_{\alpha,L}^* = \pi_{\alpha,L} + \dfrac{c_\alpha B_{3,L}^2}{2}\left(1-\left(\dfrac{\rho_L^*}{\rho_L}\right)^2\right) + a_L^2 c_\alpha\left(\dfrac{1}{\rho_L}-\dfrac{1}{\rho_L^*}\right), \\ \pi_{\alpha,R}^* = \pi_{\alpha,R} + \dfrac{c_\alpha B_{3,R}^2}{2}\left(1-\left(\dfrac{\rho_R^*}{\rho_R}\right)^2\right) + a_R^2 c_\alpha\left(\dfrac{1}{\rho_R}-\dfrac{1}{\rho_R^*}\right), \end{cases}$$

$$\begin{cases} \varepsilon_{\alpha,L}^* = & \varepsilon_{\alpha,L} + \dfrac{B_{3,L}^2}{2\rho_L}\left(1-\dfrac{\rho_L^*}{\rho_L}\right) + \dfrac{\left(\pi_{\alpha,L}^* + c_\alpha {B_{3,L}^*}^2/2\right)^2}{2(c_\alpha a_L)^2} \\ & \qquad - \dfrac{\left(\pi_{\alpha,L} + c_\alpha B_{3,L}^2/2\right)^2}{2(c_\alpha a_L)^2}, \\ \varepsilon_{\alpha,R}^* = & \varepsilon_{\alpha,R} + \dfrac{B_{3,R}^2}{2\rho_R}\left(1-\dfrac{\rho_R^*}{\rho_R}\right) + \dfrac{\left(\pi_{\alpha,R}^* + c_\alpha {B_{3,R}^*}^2/2\right)^2}{2(c_\alpha a_R)^2} \\ & \qquad - \dfrac{\left(\pi_{\alpha,R} + c_\alpha B_{3,R}^2/2\right)^2}{2(c_\alpha a_R)^2}. \end{cases} \tag{3.26}$$

Finally, using previous formulas one can compute the speeds

$$\Sigma_1 = u_{1,L} - \frac{a_L}{\rho_L}, \quad \Sigma_2 = u_1^*, \quad \Sigma_3 = u_{1,R} + \frac{a_R}{\rho_R}. \tag{3.27}$$

Remark 3.2. *Let us notice that $c_i\pi_e - c_e\pi_i$ is equal to $c_i w_{1,e} - c_e w_{1,i}$. As a consequence, $c_i\pi_e - c_e\pi_i$ is a Riemann invariant for both extreme eigenvalues. This means that this quantity remains constant through the related contact discontinuities, so that $u\partial_x(c_i\pi_e - c_e\pi_i) = 0$ there. For the central discontinuity, u is constant so that $u\partial_x(c_i\pi_e - c_e\pi_i) = \partial_x\left(u\left(c_i\pi_e - c_e\pi_i\right)\right)$ this product is also well defined in the usual weak sense.*

3.2 Numerical fluxes

All components of the system except $\overline{\mathcal{E}_e}$ and $\overline{\mathcal{E}_i}$ are conservative, thus classical computations give the associated numerical fluxes,

$$\begin{aligned} \mathcal{F}_L &= (\mathcal{F}^\rho, \mathcal{F}^{\rho u_1}, \mathcal{F}^{\rho u_2}, \mathcal{F}_L^{\overline{\mathcal{E}_e}}, \mathcal{F}_L^{\overline{\mathcal{E}_i}}, \mathcal{F}^{B_3}), \\ \mathcal{F}_R &= (\mathcal{F}^\rho, \mathcal{F}^{\rho u_1}, \mathcal{F}^{\rho u_2}, \mathcal{F}_R^{\overline{\mathcal{E}_e}}, \mathcal{F}_R^{\overline{\mathcal{E}_i}}, \mathcal{F}^{B_3}), \end{aligned} \tag{3.28}$$

where the conservative part involves the Riemann solution evaluated at x/t=0,

$$\begin{array}{lcl} \mathcal{F}^{\rho} & = & (\rho u)_{x/t=0}, \\ \mathcal{F}^{\rho u_1} & = & (\rho u_1^2 + \pi_e + \pi_i + B_3^2/2)_{x/t=0}, \\ \mathcal{F}^{\rho u_2} & = & (\rho u_1 u_2)_{x/t=0}, \\ \mathcal{F}^{B_3} & = & (u_1 B_3)_{x/t=0}. \end{array} \tag{3.29}$$

More explicitly (3.29) yields that the quantities between parentheses are evaluated at "L" if $\Sigma_1 \geq 0$, at "L^*" if $\Sigma_1 \leq 0 \leq \Sigma_2$, at "$R^*$" if $\Sigma_2 \leq 0 \leq \Sigma_3$, at "$R$" if $\Sigma_3 \leq 0$ (see Figure 1). As usual there is no ambiguity when equality occurs in these conditions.

We complete these formulas by computing the left/right numerical fluxes from (3.14) for the variables $\overline{\mathcal{E}_\alpha}$ with $\alpha = e,i$,

$$\begin{aligned} \mathcal{F}_L^{\overline{\mathcal{E}_\alpha}} = & \left(u_1\left(\overline{\mathcal{E}_\alpha} + \pi_\alpha + c_\alpha B_3^2/2\right)\right)_L + \min(0,\Sigma_1)\left(\overline{\mathcal{E}}^*_{\alpha,L} - \overline{\mathcal{E}}_{\alpha,L}\right) \\ & + \min(0,\Sigma_2)\left(\overline{\mathcal{E}}^*_{\alpha,R} - \overline{\mathcal{E}}^*_{\alpha,L}\right) + \min(0,\Sigma_3)\big(\overline{\mathcal{E}}_{\alpha,R} - \overline{\mathcal{E}}^*_{\alpha,R}\big), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \mathcal{F}_R^{\overline{\mathcal{E}_\alpha}} = & \left(u_1\left(\overline{\mathcal{E}_\alpha} + \pi_\alpha + c_\alpha B_3^2/2\right)\right)_R - \max(0,\Sigma_1)\left(\overline{\mathcal{E}}^*_{\alpha,L} - \overline{\mathcal{E}}_{\alpha,L}\right) \\ & - \max(0,\Sigma_2)\left(\overline{\mathcal{E}}^*_{\alpha,R} - \overline{\mathcal{E}}^*_{\alpha,L}\right) - \max(0,\Sigma_3)\big(\overline{\mathcal{E}}_{\alpha,R} - \overline{\mathcal{E}}^*_{\alpha,R}\big). \end{aligned} \tag{3.31}$$

3.2.1 Positivity of density and internal energies

We must now provide some sufficient conditions on a_L and a_R in order to satisfy (3.21) and the realisability of the intermediate states, that is the positivity of ρ^*_L, ρ^*_R, $\varepsilon^*_{e,L}$, $\varepsilon^*_{i,L}$, $\varepsilon^*_{e,R}$ and $\varepsilon^*_{i,R}$. First, using (3.27), (3.25), we obtain that

$$a_L(a_L + a_R) \geq \rho_L \left(a_R(u_{1,L} - u_{1,R}) + \pi_R - \pi_L\right),$$

$$a_R(a_L + a_R) \geq \rho_R \left(a_L(u_{1,L} - u_{1,R}) + \pi_L - \pi_R\right),$$

are sufficient conditions to obtain (3.22) and thus are sufficient conditions to preserve the positivity of ρ^*_L and ρ^*_R.
Second, from a straightforward calculation using (3.26), $\varepsilon^*_{e,L}$, $\varepsilon^*_{i,L}$, $\varepsilon^*_{e,R}$ and $\varepsilon^*_{i,R}$ are positive if

$$a_L \geq \max\left(\frac{|\pi_{e,L} + c_e B_{3,L}^2/2|}{2c_e\sqrt{\varepsilon_{e,L}}}, \frac{|\pi_{i,L} + c_i B_{3,L}^2/2|}{2c_i\sqrt{\varepsilon_{i,L}}}\right),$$

$$a_R \geq \max\left(\frac{|\pi_{e,R} + c_e B_{3,R}^2/2|}{2c_e\sqrt{\varepsilon_{e,R}}}, \frac{|\pi_{i,R} + c_i B_{3,R}^2/2|}{2c_i\sqrt{\varepsilon_{i,R}}}\right).$$

4 Entropy minimum principle

The present section is devoted to prove that the scheme (3.13) - (3.15) satisfies the following discrete entropy inequality

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x}\left(G(U_i,U_{i+1}) - G(U_{i-1},U_i)\right) \leq 0, \tag{4.1}$$

where

$$\eta(U) = -\rho(s_e(U) + s_i(U)), \tag{4.2}$$

is the entropy from (1.8). The numerical entropy flux $G(U_l,U_r)$ satisfies the consistency condition $G(U,U) = -\rho u\,(s_e(U) + s_i(U))$.

The state vector U is defined by (1.10) and belongs to the admissible state space Ω defined as follows

$$\Omega = \left\{U \in \mathbb{R}^6; \quad \rho > 0, \quad \varepsilon_e > 0, \quad \varepsilon_i > 0\right\}.$$

First we recall the classical and most general condition from Harten-Lax-van Leer [20] concerning discrete entropy inequalities. Let $\eta(U)$ be the entropy defined by (4.2). Let $R(\xi, U_L, U_R)$ be the approximate Riemman solver defined in Section 3.1.1. Under CFL condition (3.12), assume the following entropy consistency condition:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \eta\left(R(x/\Delta t, U_L, U_R)\right) dx \leq \frac{1}{2}\left(\eta(U_L) + \eta(U_R)\right) - \frac{\Delta t}{\Delta x}\left(\eta(U_R)u_R + \eta(U_L)u_L\right). \tag{4.3}$$

Then the scheme (3.13) - (3.15) satisfies the discrete entropy inequality (4.1). We skip the proof of this well-know result (for instance, see [20]).

The previous result states a criterion which is too weak to be applied on our scheme. Instead we use a stronger condition, based on a local minimum entropy principle. This idea was introduced in [9] for the gas dynamic sytem. A suitable extension of this technique has been derived for the MHD system in [11] and in [8] for the ten-moments system.

The first step consists in stating a sufficient condition on the intermediate states of the approximate Riemann solver to enforce the required discrete entropy inequality.

Theorem 4.1. *Assume the intermediate states U_L^* and U_R^* defined by belong Ω for all $U_{L,R} \in \Omega$ and satisfy the following entropy estimates for $\alpha = e,i$*

$$s_\alpha(U_L^*) \geq s_{\alpha,L}, \quad s_\alpha(U_R^*) \geq s_{\alpha,R}, \tag{4.4}$$

where

$$s_{\alpha,L} = s_\alpha(U_L), \quad s_{\alpha,R} = s_\alpha(U_R),$$

and $s_\alpha(U)$ is given by (1.9). *In addition assume the CFL condition* (3.12) *holds. Let $U_i^n \in \Omega$ for all $i \in \mathbb{Z}$. Then U_i^{n+1} defined by* (3.13)-(3.14) *satisfies the discrete entropy inequality* (4.1).

Remark 4.2. *The result remains valid when $B_3 = 0$ and proves that the scheme developed by a Suliciu relaxation approach in [2] also satisfies a discrete entropy inequality.*

Proof. Arguing similarly to the proof of the Theorem 6.1 in [8], the scheme will be proved to satisfy a discrete entropy inequaliy as soon as we have the following entropy estimates

$$\eta(U_L^*) \geq \eta(U_L), \quad \eta(U_R^*) \geq \eta(U_R),$$

and the following relations are satisfied

$$\rho_L^* u_1^* - \rho_L u_{1,L} = \Sigma_1 \left(\rho_L^* - \rho_L\right),$$

$$\rho_R^* u_1^* - \rho_R u_{1,R} = \Sigma_3 \left(\rho_R^* - \rho_R\right).$$

Using (4.2), we sum (4.4) for $\alpha = e, i$ and we obtain the entropy estimate. Moreover, these previous relations are Rankine-Hugoniot relations which are satisfied because the solver resolves exactly the equation on density (3.1). □

Our goal is now to obtain the sufficient condition (4.4). In the sequel, we denote $\Sigma = (\tau, \varepsilon_\alpha, B_3, \pi)^T \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ and we set $\Sigma^{eq} = (\tau, \varepsilon_\alpha, B_3, p_\alpha(\tau, \varepsilon_\alpha))^T$. Moreover we introduce three functions $\Sigma \mapsto \phi(\Sigma)$, $\Sigma \mapsto \varphi(\Sigma)$, $\Sigma \mapsto \psi(\Sigma)$ in $C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$ associated to the Riemann invariants already exhibited in (3.23), as follows

$$\begin{aligned}
&\phi(\Sigma) = \pi + c_\alpha B_3^2/2 + c_\alpha a^2 \tau, \\
&\varphi(\Sigma) = \varepsilon_\alpha + \tau B_3^2 - \frac{\left(\pi + c_\alpha B_3^2/2\right)^2}{2\left(c_e a\right)^2}, \\
&\psi(\Sigma) = \tau B_3,
\end{aligned} \tag{4.5}$$

where we have set τ the specific volume as follows

$$\tau = \frac{1}{\rho}.$$

We now give our central technical statement.

Proposition 4.3. *Let U defined by* (1.10)*. Assume that* (3.21) *holds for all $U \in \Omega$. Then there exist a function $\Sigma \mapsto \mathcal{S}_\alpha\left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)\right)$ so that*

$$\max_{\pi \in \mathbb{R}} \mathcal{S}_\alpha\left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)\right) = \mathcal{S}_\alpha\left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)\right)|_{\pi = p_\alpha(\tau, \varepsilon_\alpha)} = s_\alpha(\tau, \varepsilon_\alpha). \tag{4.6}$$

Before we give the proof of this result, let us illustrate the interest of this technical proposition, by proving Theorem 4.1.

Proof of Theorem 4.1. Here we focus on the first inequality of (4.4) since the establishment of the second inequality is similarly obtained. First we apply Proposition 4.3. Thus there exists a function $\Sigma \mapsto \mathcal{S}_\alpha(\Sigma)$ such that relation (4.6) holds. As a consequence, if we denote

$$\mathcal{S}_\alpha^*(\pi) = \mathcal{S}_\alpha^*\left(\phi(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi), \varphi(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi), \psi(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi)\right)$$

then we have

$$s_\alpha(\tau_L^*, \varepsilon_{\alpha,L}^*) = \max_{\pi \in \mathbb{R}} \mathcal{S}_\alpha^*(\pi) \geq \mathcal{S}_\alpha^*(\widetilde{\pi}), \quad \forall \widetilde{\pi} \in \mathbb{R}.$$

We fix $\widetilde{\pi} = \pi_{\alpha,L}^*$ to write

$$s_\alpha(\tau_L^*, \varepsilon_{\alpha,L}^*) \geq \mathcal{S}_\alpha^*(\pi_L^*). \tag{4.7}$$

Next the functions ϕ, φ, ψ defined by (4.5) are Riemann invariants shared by the eigenvalues $u_1 \pm a/\rho$. Thus they are invariant across the wave speeds $u_{1,L} - a_L/\rho_L$ and $u_{1,R} + a_R/\rho_R$. So we have

$$\begin{aligned} \varphi\left(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi_{\alpha,L}^*\right) &= \varphi\left(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, \pi_{\alpha,L}\right), \\ \phi\left(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi_{\alpha,L}^*\right) &= \phi\left(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, \pi_{\alpha,L}\right), \\ \psi\left(\tau_L^*, \varepsilon_{\alpha,L}^*, B_{3,L}^*, \pi_{\alpha,L}^*\right) &= \psi\left(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, \pi_{\alpha,L}\right). \end{aligned} \tag{4.8}$$

Last three equations imply that

$$\begin{aligned} &\mathcal{S}_\alpha^*(\pi_L^*) \\ &= \mathcal{S}_\alpha\left(\phi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \varphi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \psi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L)\right). \end{aligned}$$

Using last equality in (4.7) we get

$$\begin{aligned} &s_\alpha(\tau_L^*, \varepsilon_{\alpha,L}^*) \\ &\geq \mathcal{S}_\alpha\left(\phi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \varphi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \psi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L)\right). \end{aligned}$$

In addition, using second equation in (4.6), we have

$$\begin{aligned} &s_\alpha(\tau_L, \varepsilon_{\alpha,L}) \\ &= \mathcal{S}_\alpha\left(\phi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \varphi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L), \psi(\tau_L, \varepsilon_{\alpha,L}, B_{3,L}, p_L)\right). \end{aligned}$$

Thus the expected left minimum principle is reached and the proof is complete. □

To complete this section, we now establish Proposition 4.3. To address such an issue, we need the following lemma whose helpfulness is just technical. In the sequel we set $\sigma = (\tau, \varepsilon)^T$.

Lemma 4.4. *Let U defined by* (1.10) *and $\phi(\Sigma)$, $\varphi(\Sigma)$, $\psi(\Sigma)$ by* (4.5). *Assume that* (3.21) *holds for all $U \in \Omega$. Then there exists three functions, denoted by $\bar{\tau}(X,Y,Z)$, $\bar{\varepsilon}_\alpha(X,Y,Z)$, $\bar{B}_3(X,Y,Z)$ such that*

$$\bar{\tau}(\phi(\Sigma^{eq}), \varphi(\Sigma^{eq}), \psi(\Sigma^{eq})) = \tau, \quad \bar{\varepsilon}_\alpha(\phi(\Sigma^{eq}), \varphi(\Sigma^{eq}), \psi(\Sigma^{eq})) = \varepsilon \quad \text{and} \quad \bar{B}_3(\phi(\Sigma^{eq}), \varphi(\Sigma^{eq}), \psi(\Sigma^{eq})) = B_3, \tag{4.9}$$

and the following derivatives are satisfied:

$$\frac{\partial \bar{\tau}}{\partial X}(X,Y,Z) = \frac{1}{D(\tilde{\sigma})}\left(\bar{\tau} - \frac{\gamma - 1}{c_\alpha a^2}\left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right), \tag{4.10}$$

$$\frac{\partial \bar{\tau}}{\partial Y}(X,Y,Z) = \frac{-c_\alpha(\gamma - 1)}{D(\tilde{\sigma})}, \tag{4.11}$$

$$\frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(X,Y,Z) = \frac{1}{D(\tilde{\sigma})}\left(\tau\frac{\bar{B}_3^2}{2} - \frac{p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \bar{B}_3^2}{(c_\alpha a)^2}\left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right), \tag{4.12}$$

$$\frac{\partial \bar{\varepsilon}_\alpha}{\partial Y}(X,Y,Z) = \frac{1}{D(\tilde{\sigma})}\left(\bar{\tau} a^2 - p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) - c_\alpha \bar{B}_3^2\right), \tag{4.13}$$

$$\frac{\partial \bar{B}_3}{\partial X}(X,Y,Z) = \frac{-\bar{B}_3}{D(\tilde{\sigma})}\left(1 - \frac{\gamma - 1}{\bar{\tau} c_\alpha a^2}\left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right), \tag{4.14}$$

$$\frac{\partial \bar{B}_3}{\partial Y}(X,Y,Z) = \frac{c_\alpha(\gamma - 1)\bar{B}_3}{\bar{\tau} D(\tilde{\sigma})}, \tag{4.15}$$

where we have set

$$\tilde{\sigma} = (\bar{\tau}(X,Y,Z), \bar{\varepsilon}_\alpha(X,Y,Z), \bar{B}_3(X,Y,Z))^T$$

and

$$D(\tau,\varepsilon_\alpha,B_3) = c_\alpha \tau a^2 - \gamma_\alpha p_\alpha(\tau,\varepsilon_\alpha) - c_\alpha B_3^2. \tag{4.16}$$

In the second statement, we exhibit a suitable function derived from $\bar{\tau}$ and $\bar{\varepsilon}$ and we consider its local extremum.

Lemma 4.5. *Let U defined by* (1.10) *and $\phi(\Sigma)$, $\varphi(\Sigma)$, $\psi(\Sigma)$ by* (4.5)*. Assume that* (3.21) *holds for all $U \in \Omega$. We introduce $S(\Sigma) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:*

$$S(\Sigma) = s_\alpha \left(\bar{\tau} \left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma) \right), \bar{\varepsilon}_\alpha \left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma) \right) \right), \tag{4.17}$$

where $s_\alpha(\tau,\varepsilon_\alpha)$ denotes the specific entropy. For sake of clarity we denote $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$ instead of $\bar{\tau}\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$, $\bar{\varepsilon}_\alpha \left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$, and $\bar{B}_3 \left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$. Then we have

$$\frac{\partial S}{\partial \pi}(\Sigma) = \frac{\left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + \bar{B}_3^2/2\right) - \left(\pi + B_3^2/2\right)}{a^2 \bar{\varepsilon}_\alpha}, \tag{4.18}$$

with the functions $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$ defined in Lemma 4.4.

The last result concerns the study of the extrema of the function S defined by (4.17).

Lemma 4.6. *Let U defined by* (1.10) *and $S(\Sigma)$ defined by* (4.17)*. Assume that* (3.21) *holds for all $U \in \Omega$. Then the function $\pi \mapsto S(\Sigma)$ admits a unique maximum given by $\pi = p_\alpha(\tau,\varepsilon_\alpha)$.*

Equipped with these results, we can establish Proposition 4.3

Proof of Proposition 4.3. From assumption (3.21), we can apply Lemmas 4.4 and 4.5 to define a function $\mathcal{S}$ as follows:

$$\begin{aligned}\mathcal{S}(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)) &= \mathcal{S}(\Sigma), \\ &= s\left(\bar{\tau}\left(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)\right), \bar{\varepsilon}_\alpha\left(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)\right)\right),\end{aligned}$$

where the function s is nothing but the specific entropy. Now, by definition of $\bar{\tau}$ and $\bar{\varepsilon}_\alpha$, we have

$$\bar{\tau}\left(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)\right)|_{\pi=p(\tau,\varepsilon_\alpha)} = \tau \text{ and } \bar{\varepsilon}_\alpha\left(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)\right)|_{\pi=p(\tau,\varepsilon_\alpha)} = \varepsilon_\alpha,$$

which immediately implies

$$\mathcal{S}\left(\phi(\Sigma),\varphi(\Sigma),\psi(\Sigma)\right)|_{\pi=p(\tau,\varepsilon_\alpha)} = s(\tau,\varepsilon_\alpha).$$

Next, we study the extrema of the function $\pi \mapsto S(\Sigma)$. By Lemma 4.6, the unique maximum of the function S is $\pi = p_\alpha(\tau,\varepsilon_\alpha)$, which completes the proof. □

We conclude this section by giving successively the proof of the three intermediate lemmas.

Proof of Lemma 4.4. Let us consider the function

$$\Theta : (\tau, \varepsilon_\alpha, B_3) \mapsto \begin{pmatrix} \phi(\tau, \varepsilon_\alpha, B_3, p_\alpha(\tau, \varepsilon_\alpha)) \\ \varphi(\tau, \varepsilon_\alpha, B_3, p_\alpha(\tau, \varepsilon_\alpha)) \\ \psi(\tau, \varepsilon_\alpha, B_3, p_\alpha(\tau, \varepsilon_\alpha)) \end{pmatrix}. \tag{4.19}$$

We remark that the function $D(\tau,\varepsilon_\alpha,B_3)$, defined by (4.16) is nothing but the Jacobian function of Θ. Since for all $(\tau,\varepsilon_\alpha,B_3)$ under consideration (3.21) holds, $D(\tau,\varepsilon_\alpha,B_3)$ does not vanish. Thus we can apply the inverse function theorem to deduce the existence of a reciprocal function

$$\Theta^{-1}(X, Y, Z) = \begin{pmatrix} \bar{\tau}(X,Y,Z) \\ \bar{\varepsilon}_\alpha(X,Y,Z) \\ \bar{B}_3(X,Y,Z) \end{pmatrix},$$

defined for (X,Y,Z) in the range of Θ and such that $D(\tau,\varepsilon_\alpha,B_3) \neq 0$.
By definition of the functions $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$, the relation (4.9) is obviously obtained.
Now, we evaluate the derivatives of those reciprocal functions. Once again by definition of $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$, we have

$$\begin{aligned} &\varphi\left(\bar{\tau}(X,Y,Z), \bar{B}_3(X,Y,Z), p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)\right) = X, \\ &\phi\left(\bar{\tau}(X,Y,Z), \bar{\varepsilon}_\alpha(X,Y,Z), \bar{B}_3(X,Y,Z), p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)\right) = Y, \\ &\psi\left(\bar{\tau}(X,Y,Z), \bar{B}_3(X,Y,Z)\right) = Z. \end{aligned}$$

By differentiating in X these two relations and using (4.5), we obtain:

$$\begin{aligned} \left(c_\alpha a^2 - \frac{p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)}{\bar{\tau}}\right) \frac{\partial \bar{\tau}}{\partial X} + \frac{c_\alpha(\gamma - 1)}{\bar{\tau}} \frac{\partial \bar{\varepsilon}_\alpha}{\partial X} + c_\alpha \bar{B}_3 \frac{\partial \bar{B}_3}{\partial X} \qquad &= 1, \\ \left(\frac{\bar{B}_3^2}{2} + \frac{p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)}{(c_\alpha a)^2 \bar{\tau}} \left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right) \frac{\partial \bar{\tau}}{\partial X} & \\ + \left(1 - \frac{p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)}{(c_\alpha a)^2 \bar{\varepsilon}_\alpha} \left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right) \frac{\partial \bar{\varepsilon}_\alpha}{\partial X} & \\ + \bar{B}_3 \left(\bar{\tau} - \frac{1}{c_\alpha a^2} \left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right) \frac{\partial \bar{B}_3}{\partial X} \qquad &= 0, \\ \bar{B}_3 \frac{\partial \bar{\tau}}{\partial X} + \bar{\tau} \frac{\partial \bar{B}_3}{\partial X} \qquad &= 0. \end{aligned}$$

With $D(\tau,\varepsilon_\alpha,B_3) \neq 0$, this above 3×3 system is solvable in the variables $(\frac{\partial \bar{\tau}}{\partial X}, \frac{\partial \bar{\varepsilon}_\alpha}{\partial X}, \frac{\partial \bar{B}_3}{\partial X})$. Then we get the expected definition of the derivatives $\frac{\partial \bar{\tau}}{\partial X}(X,Y,Z)$, $\frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(X,Y,Z)$ and $\frac{\partial \bar{B}_3}{\partial X}(X,Y,Z)$ given by (4.10), (4.12), and (4.14). □

Proof of Lemma 4.5. Here, we have to compute the derivative of S with respect to π where the function S is defined by (4.17). Using the definition of ψ in (4.5) we notice that $\frac{\partial \psi}{\partial \pi}(\Sigma) = 0$, which enables us to write

$$\begin{aligned} \frac{\partial S}{\partial \pi}(\Sigma) \quad = \quad & \frac{\partial s_\alpha}{\partial \tau}(\bar{\sigma}) \left(\frac{\partial \bar{\tau}}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \varphi}{\partial \pi}(\Sigma)\right. \\ & \qquad \left. + \frac{\partial \bar{\tau}}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \phi}{\partial \pi}(\Sigma)\right) \\ + \quad & \frac{\partial s_\alpha}{\partial \varepsilon_\alpha}(\bar{\sigma}) \left(\frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \varphi}{\partial \pi}(\Sigma)\right. \\ & \qquad \left. + \frac{\partial \bar{\varepsilon}_\alpha}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \phi}{\partial \pi}(\Sigma)\right), \end{aligned} \tag{4.20}$$

where we have set

$$\bar{\sigma} = \left(\bar{\tau}\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right), \bar{\varepsilon}_\alpha\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)\right)^T.$$

In the sequel, we will denote $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$ instead of $\bar{\tau}\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$, $\bar{\varepsilon}_\alpha\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$ and $\bar{B}_3\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)$. Moreover we will use the notation $\tilde{\sigma} = \left(\bar{\tau}, \bar{\varepsilon}_\alpha, \bar{B}_3\right)$. Next using (4.10)-(4.13), we compute

$$\begin{aligned}\frac{\partial \bar{\tau}}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma))\frac{\partial \varphi}{\partial \pi}(\Sigma) + \frac{\partial \bar{\tau}}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma))\frac{\partial \phi}{\partial \pi}(\Sigma) = \\ \frac{1}{D(\tilde{\sigma})}\left(\bar{\tau} + \frac{(\gamma-1)}{c_\alpha a^2}\left[\left(\pi + c_\alpha \frac{B_3^2}{2}\right) - \left(p_\alpha(\bar{\tau},\bar{\varepsilon}) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right]\right), \end{aligned} \quad (4.21)$$

and

$$\begin{aligned}\frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma))\frac{\partial \varphi}{\partial \pi}(\Sigma) + \frac{\partial \bar{\varepsilon}_\alpha}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma))\frac{\partial \phi}{\partial \pi}(\Sigma) = \\ \frac{1}{D(\tilde{\sigma})}\left(\bar{\tau}\left(\frac{\bar{B}_3^2}{2} - \frac{1}{c_\alpha}\left(\pi + c_\alpha \frac{B_3^2}{2}\right)\right) + \frac{p_\alpha(\bar{\tau},\bar{\varepsilon}_\alpha) + c_\alpha \bar{B}_3^2}{(c_\alpha a)^2}\right. \\ \left.\left[\left(\pi + c_\alpha \frac{B_3^2}{2}\right) - \left(p_\alpha(\bar{\tau},\bar{\varepsilon}) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right]\right). \end{aligned} \quad (4.22)$$

Then we plug into (4.20) the two previous relations together with

$$\frac{\partial s_\alpha}{\partial \tau}(\bar{\sigma}) = \frac{(\gamma-1)}{\bar{\tau}}, \quad \text{and} \quad \frac{\partial s_\alpha}{\partial \varepsilon_\alpha}(\bar{\sigma}) = \frac{1}{\varepsilon_\alpha},$$

to obtain the following equation:

$$\frac{\partial S}{\partial \pi}(\Sigma) = \frac{1}{D(\tilde{\sigma})}\left(A + B\left[\left(\pi + c_\alpha \frac{B_3^2}{2}\right) - \left(p_\alpha(\bar{\tau},\bar{\varepsilon}) + c_\alpha \frac{\bar{B}_3^2}{2}\right)\right]\right), \quad (4.23)$$

with

$$A = (\gamma - 1) + \frac{\bar{\tau}}{c_\alpha \bar{\varepsilon}_\alpha}\left(c_\alpha \frac{\bar{B}_3^2}{2} - \pi - c_\alpha \frac{B_3^2}{2}\right), \; B = \frac{(\gamma-1)^2}{c_\alpha a^2 \bar{\tau}} + \frac{p_\alpha(\bar{\tau},\bar{\varepsilon}) + c_\alpha \bar{B}_3^2}{(c_\alpha a)^2 \bar{\varepsilon}_\alpha}.$$

We now write

$$A = \frac{\bar{\tau}}{c_\alpha \bar{\varepsilon}_\alpha}\left(p_\alpha(\bar{\tau},\bar{\varepsilon}_\alpha) + c_\alpha \frac{\bar{B}_3^2}{2} - \pi - c_\alpha \frac{B_3^2}{2}\right), \quad B = \frac{\gamma p_\alpha(\bar{\tau},\bar{\varepsilon}_\alpha) + c_\alpha \bar{B}_3^2}{(c_\alpha a)^2 \bar{\varepsilon}_\alpha \bar{\tau}}.$$

Finally we plug these equalities into (4.23) which gives (4.18). □

Proof of Lemma 4.6. We apply Lemma 4.4 and Lemma 4.5, which enables us to introduce the function

$$g(\Sigma) = \left(p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + \bar{B}_3^2/2\right) \circ (\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)) - \left(\pi + B_3^2/2\right),$$

where $\Sigma = (\tau, \varepsilon_\alpha, B_3, \pi)^T$.
In order to prove the lemma, we are going to study the roots of the function g. First we notice that, thanks to relations in (4.9), $\pi^0 = p_\alpha(\tau, \varepsilon_\alpha)$ is a root of g. Then we will show that for every $\tilde{\pi}^0$ root of g, we have $\frac{\partial g}{\partial \pi}(\tilde{\pi}^0) < 0$.

Let $\tilde{\pi}^0$ be a root of g. We denote by g' the partial derivative of g with respect to π, and we compute it as follows:

$$
\begin{aligned}
g'(\Sigma) \quad = \quad & \frac{\partial p_\alpha}{\partial \tau}(\bar{\sigma}) \left(\frac{\partial \bar{\tau}}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \varphi}{\partial \pi}(\Sigma) \right. \\
& \qquad \left. + \frac{\partial \bar{\tau}}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \phi}{\partial \pi}(\Sigma) \right) \\
& + \frac{\partial p_\alpha}{\partial \varepsilon_\alpha}(\bar{\sigma}) \left(\frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \varphi}{\partial \pi}(\Sigma) \right. \\
& \qquad \left. + \frac{\partial \bar{\varepsilon}_\alpha}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \phi}{\partial \pi}(\Sigma) \right) \\
& + \bar{B}_3(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \left(\frac{\partial \bar{B}_3}{\partial X}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \varphi}{\partial \pi}(\Sigma) \right. \\
& \qquad \left. + \frac{\partial \bar{B}_3}{\partial Y}(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)) \frac{\partial \phi}{\partial \pi}(\Sigma) \right) - 1,
\end{aligned}
\tag{4.24}
$$

where

$$\bar{\sigma} = \left(\bar{\tau}\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right), \bar{\varepsilon}_\alpha\left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)\right)^T.$$

In the sequel, we use the notation $\Sigma^0 = (\tau, \varepsilon_\alpha, B_3, \tilde{\pi}^0)$ and we are interested in computing $g'(\Sigma^0)$. For sake of clarity we will denote $\bar{\tau}$, $\bar{\varepsilon}_\alpha$ and $\bar{B}_3$ instead of $\bar{\tau}\left(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)\right)$, $\bar{\varepsilon}_\alpha\left(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)\right)$ and $\bar{B}_3\left(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)\right)$. Moreover we will denote $\tilde{\sigma} = \left(\bar{\tau}, \bar{\varepsilon}_\alpha, \bar{B}_3)\right)^T$.

First we use the previous computations (4.21), (4.22) and the fact that $g(\Sigma^0) = 0$ to get

$$
\begin{aligned}
& \frac{\partial \bar{\tau}}{\partial X}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \varphi}{\partial \pi}(\Sigma^0) \\
& \qquad + \frac{\partial \bar{\tau}}{\partial Y}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \phi}{\partial \pi}(\Sigma^0) = \frac{\bar{\tau}}{D(\tilde{\sigma})},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\partial \bar{\varepsilon}_\alpha}{\partial X}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \varphi}{\partial \pi}(\Sigma^0) \\
& \qquad + \frac{\partial \bar{\varepsilon}_\alpha}{\partial Y}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \phi}{\partial \pi}(\Sigma^0) = \frac{-\bar{\tau} p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha)}{c_\alpha D(\tilde{\sigma})}.
\end{aligned}
$$

Moreover combining $g(\Sigma^0) = 0$ and relations (4.5), (4.14), (4.15) we get that

$$
\begin{aligned}
& \frac{\partial \bar{B}_3}{\partial X}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \varphi}{\partial \pi}(\Sigma^0) \\
& \qquad + \frac{\partial \bar{B}_3}{\partial Y}(\varphi(\Sigma^0), \phi(\Sigma^0), \psi(\Sigma^0)) \frac{\partial \phi}{\partial \pi}(\Sigma^0) = \frac{-c_\alpha \bar{B}}{D(\tilde{\sigma})}.
\end{aligned}
$$

Then we plug in (4.24) last three previous relations together with

$$\frac{\partial p_\alpha}{\partial \tau}(\bar{\sigma}) = \frac{-p_\alpha}{\tau} \quad \text{and} \quad \frac{\partial s_\alpha}{\partial \varepsilon_\alpha}(\bar{\sigma}) = \frac{p_\alpha}{\varepsilon_\alpha},$$

in order to get

$$g'(\Sigma^0) = -\frac{\gamma p_\alpha(\bar{\tau}, \bar{\varepsilon}_\alpha) + c_\alpha \bar{B}_3 + D(\tilde{\sigma})}{D(\tilde{\sigma})}.$$

Thus $g'(\Sigma^0) < 0$ for all roots g. The function g being continuous, $\pi \mapsto g(\Sigma)$ has a unique root, which is $\pi = p_\alpha(\tau, \varepsilon_\alpha)$.
To complete the proof we will show that $\pi = p_\alpha(\tau, \varepsilon_\alpha)$ is a maximum of the function S in Lemma 4.5. Indeed we can rewrite (4.18) as

$$\frac{\partial S}{\partial \pi}(\Sigma) = \frac{g(\Sigma)}{a^2 \bar{\varepsilon}_\alpha \left(\varphi(\Sigma), \phi(\Sigma), \psi(\Sigma)\right)}.$$

Since $\pi = p_\alpha(\tau, \varepsilon_\alpha)$ is a unique root of g, $g(\Sigma^{eq}) = 0$ and from the previous relation we deduce that $\frac{\partial S}{\partial \pi}(\Sigma^{eq}) = 0$. Moreover we have

$$\frac{\partial^2 S}{\partial \pi^2}(\Sigma^{eq}) = \frac{g'(\Sigma^{eq})}{a^2 \bar{\varepsilon}_\alpha \left(\varphi(\Sigma^{eq}), \phi(\Sigma^{eq}), \psi(\Sigma^{eq})\right)},$$

and we have previously shown that g' is negative for all roots of g. So we finally get that $\frac{\partial S}{\partial \pi}(\Sigma^{eq}) = 0$ and $\frac{\partial^2 S}{\partial \pi^2}(\Sigma^{eq}) < 0$. Thus $\pi = p_\alpha(\tau, \varepsilon_\alpha)$ is the maximum of the function S, which concludes the proof. □

5 Numerical tests

In this section we perform numerical approximations in order to evaluate both accuracy and robustness of the scheme. First order in time and space are evaluated. The CFL number is 1/4 in all tests. The computations are performed over interval [0,1], until time $t = 0.1$. For all test case we use the following values for the parameters

$$k_B = 1, \quad \gamma_e = \gamma_i = \frac{5}{3}, \quad Z = 1, \quad m_e = 10^{-3}, \quad m_i = 1, \quad u_2 = 0.$$

The numerical test are organized as follows. The subsection 5.1 is dedicated to the approximation of the homogenous system via two test cases. First we investigate the solution of a Riemann problem involving continuous and discontinuous waves in order to observe the qualitative behavior of the numerical solution. Second we considerer a robustness test case dedicated to a double rarefaction wave. Then the subsection 5.2 is dedicated to the approximation of the source term using an analytical solution.

5.1 Resolution of Riemann problems

Here we solve the homogeneous part of the system (1.1) - (1.6) . We consider two test cases. The first one corresponds to a classical shock tube Sod problem whereas the second one is a symmetric rarefaction waves problem. The left and right states of the Riemann problems are written in table 1 in terms of density, velocity, transverse magnetic field, electronic and ionic temperatures.

The numerical solution for test 1 consists of, from left to right, a left rarefaction wave, a material contact and a right shock. The results are shown in Figure 2. We observe the right structure of wave and the convergence of the scheme.

The numerical solution for test 2 consists of two rarefaction waves travelling in opposite directions. The results are displayed in Figure 3 and show no oscillations in the results. This illustrates the robustness of the scheme when dealing with vacuum data.

Table 1: Data for the different test cases

	ρ_L	$u_{1,L}$	$u_{2,L}$	$B_{3,L}$	$T_{3,L}$	$T_{3,L}$
Test case 1	1	1	0	1	1	1
Test case 2	1	-10	0	0	1	1
	ρ_R	$u_{1,R}$	$u_{2,R}$	$B_{3,R}$	$T_{3,R}$	$T_{3,R}$
Test case 1	0.9	0.8	0	0.6	0.5	0.4
Test case 2	1	10	0	1	1	1

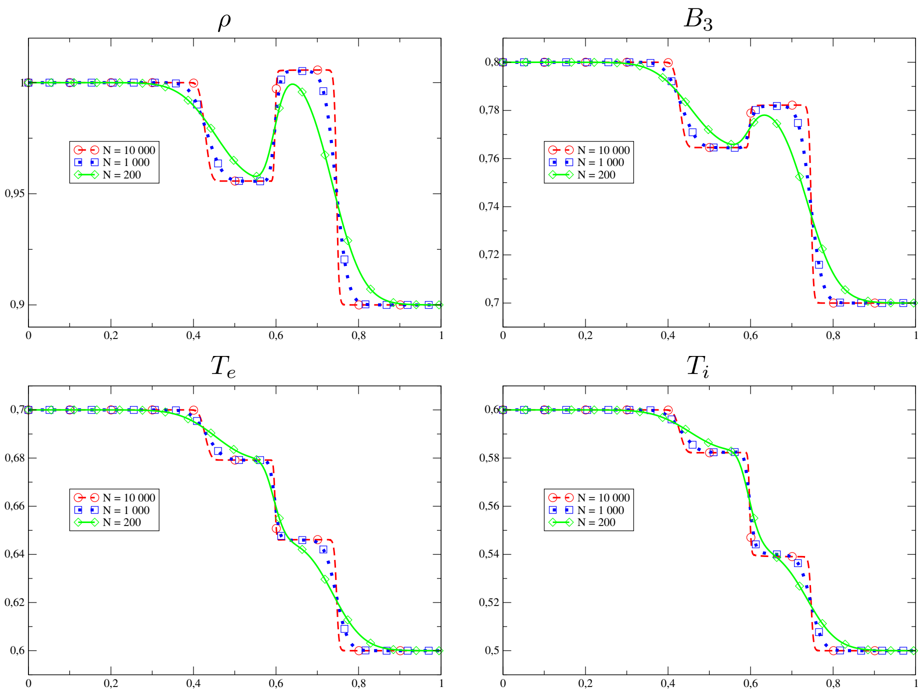

Figure 2: Solution for Test case 1 for the components ρ, B_3, T_e, T_i computed with 200, 1000 and 10 000 cells.

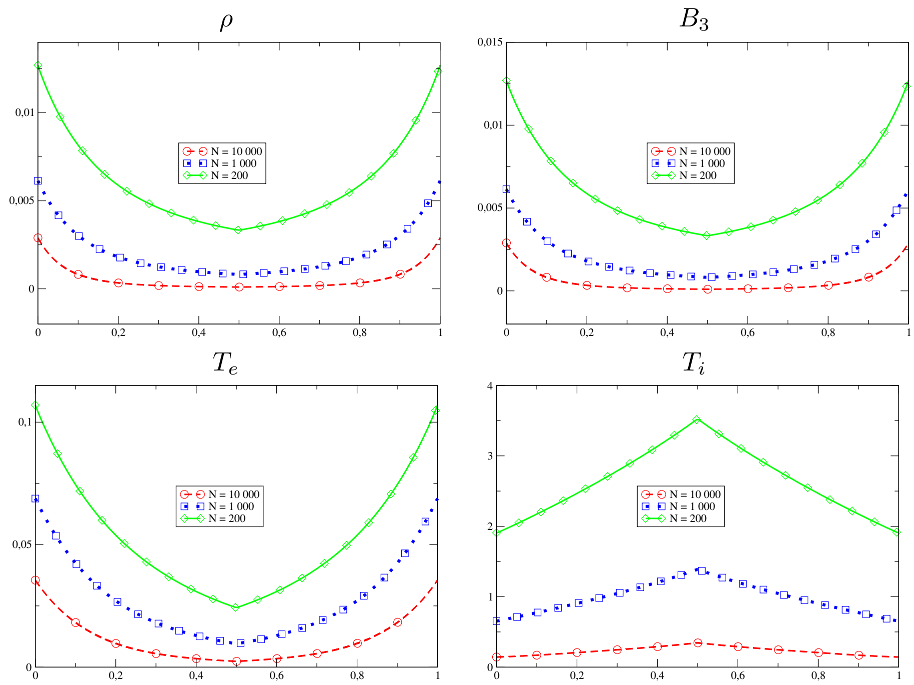

Figure 3: Solution for Test case 2 for ρ, B_3, T_e, T_i computed with 200, 1000 and 10 000 cells.

5.2 Source term approximation: an analytical solution

We take initial data such that

$$\forall x \in \mathbb{R}, \quad \rho(x,0) = 1, \quad u_1(x,0) = 10, \quad T_i(x,0) + T_e(x,0) = \overline{T} = 2.$$

An exact solution of the system (1.1)-(1.6) is available,

$$\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad \rho(x,t) = 1, \quad u_1(x,t) = 10,$$

$$T_e(x,t) = T_e(x - u_1 t, 0) e^{-2\mu t} + a(x,t)\left(1 - e^{-2\mu t}\right) + b(x,t)\left(2\mu t - 1 + e^{-2\mu t}\right), \tag{5.1}$$

where $\mu = (\gamma_\alpha - 1)\nu_{1,\alpha\beta}/(n_e k_B)$ and

$$a(x,t) = \frac{\overline{T}}{2} + \frac{\widetilde{\nu}_{2,ei}}{2\nu^1_{ei}} \left(\partial_x B_3(x - u_1 t, 0)\right)^2,$$

$$b(x,t) = \frac{\widetilde{\nu}_{2,ei} + \widetilde{\nu}_{2,ie}}{4\mu} \left(\partial_x B_3(x - u_1 t, 0)\right)^2.$$

Here we choose $\overline{T} = 2$ and for all $x \in [0,1]$,

$$T_e(x,0) = \frac{\overline{T}}{2} + \exp(-200(x - 1/2)^2), \text{ and } B_3(x,0) = \exp(-50(x - 1/2)^2).$$

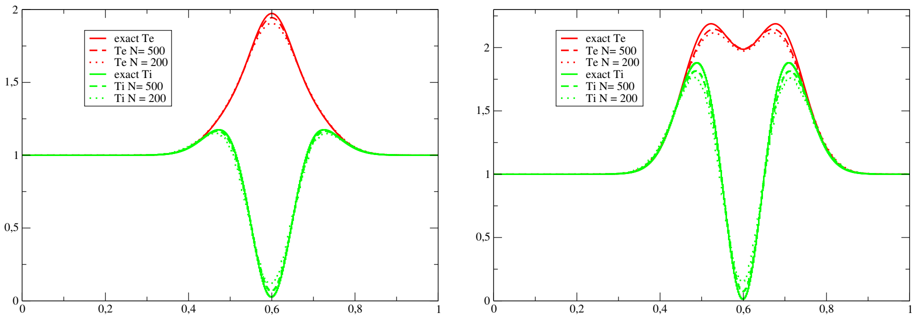

Figure 4: Solution for analytic test case for the temperatures T_e, T_i computed with 200 and 500 cells. On the left $\widetilde{\nu}_{2,ei} = 2$ and on the right $\widetilde{\nu}_{2,ei} = 4$.

Moreover we set $\nu_{1,ei} = 1$ and $\widetilde{\nu}_{2,ei} = \widetilde{\nu}_{2,ie}$. The results for $\widetilde{\nu}_{2,ei} = 2$ and $\widetilde{\nu}_{2,ei} = 4$ are displayed in figure 4. They show a good agreement with analytical solution (5.1).

6 Conclusion

In this paper we have investigated at the modelling and numerical point of view the bitemperature Euler system with transverse magnetic field.

At the modelling point of view, we introduced a multicomponent BGK kinetic system coupled with Mawell equations. Next by perfoming an hydrodynamic limit, the bitemperature Euler equations with transverse magnetic field has been established.

At the numerical level, we designed a Suliciu relaxation approximation and we showed that the associated scheme is entropic. This property should play a fundamental role for selecting the correct solutions.

We shall adress to forthcoming papers the following different points. Firstly, we are currently working on taking into account transverse electric (TE) fields. Then, we plan to use this work dealing with transverse magnetic field and future work on TE field in order to propose novel high order and multi-dimensional schemes. Moreover, the case of Navier-Stokes asymptotics has also to be considered.

Acknowledgement

The autors want to thank Denise Aregba for fruitful discussions that gave rise to this work. This work was partially granted by the Conseil Régional de la région Nouvelle-Aquitaine.

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