Two compressible immiscible fluids in porous media: The case where the porosity depends on the pressure.

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Abstract

Consider a model of flow of two compressible or incompressible and immiscible phases in a three dimensional porous media. The existence of a weak solution is obtained for two compressible immiscible fluids when the porosity depends on the global pressure and on the space variable.

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1 Introduction and presentation of the model.

As a mathematical point of view, the study of the immiscible flow models has been investigated in ([1], [2]). In ([2]), the study is performed by using the global pressure. By this approach, the models are described with one pressure variable and several saturation variables.
The case of two incompressible phases has been investigated in ([1], [2], [4], [8]). In ([10]), the authors consider the case of a mixture of a compressible phase and of a incompressible phase when the porosity is independant of the global pressure. The case of two compressible phases had been performed in ([11]) when the porosity is independent of the global pressure. In ([3]), the authors proved the existence of a weak solution for two incompressible immiscible fluids when the porosity depends on the global pressure. Here,

we obtain the existence of a solution in the situation of two compressible fluids when the porosity depends on the global pressure and on the space variable.
The equations describing the immiscible displacement of two compressible fluids are given by

$$\partial_t(\phi\rho_i s_i)(t,x) + div(\rho_i \mathbf{V}_i)(t,x) + \rho_i s_i f_P(t,x) = \rho_i s_i^I f_I(t,x), \ i = 1,2, \quad (1.1)$$

where ϕ is the porosity of the medium. ρ_i and s_i are respectively the density and the saturation of the i^{th} fluid. The velocity $\mathbf{V}_i$ of each fluid is given by the Darcy law

$$\mathbf{V}_i(t,x) = -\mathbf{K}(x)\frac{k_i(s_i(t,x))}{\mu_i}\nabla p_i(t,x), \quad i = 1,2,$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, k_i the relative permeability of the i^{th} phase, μ_i the constant i-phase's viscosity and p_i the i-phase's pressure. The effect of the gravity is neglected. The functions f_I and f_P are respectively the injection and production terms. By definition of saturations

$$s_1(t,x) + s_2(t,x) = 1. \quad (1.2)$$

Consider the capillary pressure p_{12} defined by

$$p_{12}(s_1(t,x)) = p_1(t,x) - p_2(t,x). \quad (1.3)$$

Denote that the function $s \mapsto p_{1,2}(s)$ is nondecreasing ($\frac{p_{1,2}}{ds}(s) \geq 0$ for all $s \in [0,1]$).
Therefore the unknown of the problem are the saturation of the first specy and the global pressure.
Consider now the i phase's mobility, $M_i(s_i)$, the total mobility $M(s_1)$ and the total velocity by the expressions

$$M_i(s_i) = \frac{k_i(s_i)}{\mu_i}, \ M(s_1) = M_1(s_1) + M_2(1-s_1), \ V = V_1 + V_2. \quad (1.4)$$

As in [10, 11, 2], the total velocity can be expressed in terms of p_2 and p_{12} as follows

$$\mathbf{V}(t,x) = -\mathbf{K}(x)M(s_1)\big(\nabla p_2(t,x) + \frac{M_1(s_1)}{M(s_1)}\nabla p_{12}(s_1)\big).$$

By defining a function $\tilde{p}(s_1)$ such that $\frac{d\tilde{p}}{ds}(s_1) = \frac{M_1(s_1)}{M(s_1)}\frac{dp_{12}}{ds}(s_1)$, the global pressure p writes $p = p_2 + \tilde{p}$. As in [2], V satisfies the relation

$$\mathbf{V}(t,x) = -\mathbf{K}(t,x)M(s_1)\nabla p(t,x).$$

So the expression of each phase velocity is given by

$$\mathbf{V}_i = -\mathbf{K}\, M_i(s_i)\nabla p - \mathbf{K}\alpha(s_1)\nabla s_i, \tag{1.5}$$

where

$$\alpha(s_1) = \frac{M_1(s_1)M_2(s_1)}{M(s_1)}\frac{dp_{12}}{ds}(s_1) \geq 0.$$

The density depends on the pressure of the respective fluid and the porosity depends on the space variable and on the pressure. Suppose that the density and the porosity depend only on the global pressure p. This assumption is valid if the capillary pressure p_{12} is low compared to the pressure of the gases p_1 and p_2. So we can assume that $\rho_i = \rho_i(p)$ and $\phi = \phi(x,p)$.
By taking (1.5) into account, the system (1.1, 1.5) can be transformed into

$$\begin{aligned} \partial_t(\phi\rho_i s_i)(t,x) - div(\mathbf{K}\rho_i M_i(s_i)\nabla p))(t,x) - div(\mathbf{K}\rho_i\alpha(s_1)\nabla s_i) + \rho_i s_i f_P(t,x) \\ = \rho_i s_i^I f_I(t,x),\ i = 1,2, \end{aligned} \tag{1.6}$$

with the condition (1.2).
Let $T > 0$ be fixed and Ω be a bounded set of $\mathbb{R}^d$ ($d \geq 1$). Consider the sets $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$.
The solutions to (1.1) have to satisfy the following boundary conditions. The boundary $\partial\Omega$ writes as $\partial\Omega = \Gamma_1 \cup \Gamma_{imp}$ with $mes(\Gamma_1) > 0$. Here Γ_1 denotes the injection boundary of the second phase and Γ_{imp} the imprevious one.

$$\begin{aligned} s(t,x) = 0,\ p(t,x) = 0 \text{ on } \Gamma_1, \\ \mathbf{K}\nabla p \cdot n = \mathbf{K}\alpha(s_1)\cdot n = 0 \text{ on } \Gamma_{imp}, \end{aligned} \tag{1.7}$$

where n is the outward normal to the boundary Γ_{imp}. The pressure is kept constant (shifted at zero) during the time on the region of injection.
The initial conditions for the pressure and the saturation are

$$p(0,x) = p^0(x) \text{ in } \Omega, \tag{1.8}$$

$$s_1(0,x) = s_1^0(x) \text{ in } \Omega. \tag{1.9}$$

Next we shall perform the following assumptions on the system

$(H1)$ The tensor $\mathbf{K} \in (W^{1,\infty}(\Omega))^{d\times d}$ and there are nonnegative constants k_0 and k_∞ such that

$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty$ and $\langle \mathbf{K}(x)\xi, \xi\rangle \geq k_0|\xi|^2$ for all $\xi \in \mathbb{R}^d$ $a.e$ $x \in \Omega$.

$(H2)$ The functions M_1 and $M_2 \in \mathcal{C}^0([0,T];\mathbb{R}_+)$, satisfy $M_1(s_1 = 0) = 0$ and $M_2(s_2 = 0) = 0$. Moreover, there is a nonnegative constant m_0 such that, for all $s_1 \in [0,1]$,

$$M_1(s_1) + M_1(1 - s_1) \geq m_0.$$

$(H3)$ $(f_P, f_I) \in (L^2(\Omega))^2$, f_P, $f_I \geq 0$ $a.e$ $(t,x) \in Q_T$, $s_i^I \geq 0$ $(i = 1,2)$ and $s_1^I + s_2^I = 1$ $a.e$ in $(t,x) \in Q_T$.
$(H4)$ The densities ρ_i $(i = 1,2)$ and the porosity $\phi \in \mathcal{C}^2(\mathbb{R})$, are non decreasing with respect to the variable x and there are $\rho_m > 0$, $\rho_M >$, $\phi_m > 0$, $\phi_M >$ such that $\rho_m \leq \rho_i(p) \leq \rho_M$ for all p and $\phi_m \leq \phi(x,p) \leq \phi_M$ for all p and $x \in \Omega$. Moreover $\partial_p(\phi\rho_1)$ and $\nabla(\phi\rho_1)$ are bounded.
$(H5)$ The function $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}_+)$ and there is a constant η such that $\alpha(x) \geq \eta$.
$(H6)$ The functions $k(p) = \int_0^p \nabla\phi(x,q)dq$, $k_2(p) = \int_0^p \phi(x,q)dq$ and $k_3(p) = \int_0^p \Delta(x,q)dq$ are bounded.

f_I and f_P are respectively the injection and production term. Denote that the assumption $(H6)$ is not performed in ([10, 11]) because in those papers the porosity ϕ does not depend on the p variable.
Define next

$$\beta(s) = \int_0^s \alpha(z)dz$$

and the Sobolev space

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_1\},$$

together with the norm $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega)^d)}$.
Define $g_1(x,p) = \int_0^p \phi(x,q)\rho_2(q)dq$ and $g_2(x,p) = \int_0^p \phi(x,q)\rho_1(q)dq$. The functions $\mathcal{H}_1(x,p)$ and $\mathcal{H}_2(x,p)$ defined by

$$\mathcal{H}_1(p,x) = \rho_1(p)g_1(p)\phi(p) - \int_0^p \phi(q)^2\rho_1(q)\rho_2(q)dq, \tag{1.10}$$

$$\mathcal{H}_2(p,x) = \rho_2(p)g_2(p)\phi(p) - \int_0^p \phi(q)^2\rho_1(q)\rho_2(q)dq, \tag{1.11}$$

satisfy $\frac{\partial}{\partial p}\mathcal{H}_i(x,p) = \frac{\partial}{\partial p}(\rho_i\phi)(x,p)g_i(x,p)$, $\mathcal{H}_i(0) = 0$, $\mathcal{H}_i(p) \geq 0$ for all p, and $\mathcal{H}_i$ is bounded. Multiply (1.6) taken for $i = 1$ by g_1 and (1.6) taken for $i = 2$ by g_2, add the two equations and integrate on Ω lead to

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega s_1\mathcal{H}_1(x,p)dx + \frac{d}{dt}\int_\Omega s_1\mathcal{H}_2(x,p)dx \\
+\int_\Omega \rho_1(p)\rho_2(p)\phi(x,p)(M_1(s_1)+M_2(s_2))\mathbf{K}\nabla p\cdot\nabla p\,dx \\
+\int_\Omega \mathbf{K}\nabla p\cdot\Big(\rho_1 M_1(s_1)\int_0^p \nabla\phi(x,q)\rho_2 dq - \rho_2 M_2(s_2)\cdot\int_0^p \nabla_x\phi(x,q)\rho_1 dq\Big)dx \\
+\int_\Omega \mathbf{K}\alpha_1(s_1)\,(\rho_1+\rho_2)\int_0^p \nabla\phi(x,q)\rho_2 dq\cdot\nabla s_1\,dx \\
+\int_\Omega (\rho_1(p)g_1(p)s_1 + \rho_2(p)g_2(p)s_2)f_P dx \\
= \int_\Omega (\rho_1(p)g_1(p)s_1^I + \rho_2(p)g_2(p)s_2^I)f_I dx.
\end{aligned}
$$

The main result of this paper is the following

Theorem 1.1. *Let $(H1)-(H6)$ hold. Let s_i^0, p^0 be defined almost everywhere in Ω. Then there exists (s_1,p) satisfying*

$$0 \leq s_i(t,x) \leq 1 \; a.e \; in \; Q_T,\; s_i \in L^2(0,T;H^1_{\Gamma_1}),\; \phi(p)\rho_i(p)s_i \in \mathcal{C}^0(0,T;L^2(\Omega)), i = 1,2,$$

the boundary conditions (1.7), the initial conditions (1.8, 1.9) and the weak formulation for all $\varphi \in L^2(0,T;H^1_{\Gamma_1})$,

$$
\begin{aligned}
\langle\partial_t(\phi\rho_i s_i),\varphi\rangle + \int_{Q_T}\rho_i(p)M_i(s_i)\mathbf{K}\,\nabla_x p\cdot\nabla\varphi\,dxdt \\
+\int_{Q_T}\mathbf{K}\rho_i(p)\alpha(s_1)\nabla s_i\cdot\nabla\varphi dxdt + \int_{Q_T}\rho_i(p)s_i f_P\,\varphi dxdt \\
= \int_{Q_T}\rho_i(p)s_i f_I\varphi dxdt. \qquad i = 1,2. \qquad (1.12)
\end{aligned}
$$

Remark 1. *If one of the phase is compressible, Theorem 1.1 still holds and constitutes a generalization of the results given in ([10]) in the case of one compressible phase and one incompressible phase. This is mainly due to $\partial_p\phi > 0$.*

Remark 2. *Assumption $(H5)$ avoids the degeneracies in 0 and in 1 for α. By reasoning as in ([11]), the problem of degeneracies can also been considered.*

This paper gives a generalization of the strategies developped in [10, 11]. The method is extended to the situation where the porosity depends on the global pressure p and on the space variable x. The situation when the porosity depends on the global pressure has only been considered in [3] for two incompressible flows. Denote that the case where ϕ depends only on p can be solved by arguing as in [11] by changing ρ_1 and ρ_2 into $\phi\rho_1$ and $\phi\rho_2$. This paper is organized as follows. The second section is devoted to the resolution of an elliptic system which is a discretized version of (1.6). Section 3 deals with the passage to the limit in this discretized equation.

2 Study of a nonlinear elliptic system.

As in ([11]), consider the following equation which is discretized in time,

$$\frac{(\phi\rho_i)(x,p)s_i - \phi^*\rho_i^* s_i^*}{h} - div\big(\mathbf{K}\rho_i(p)M_i(s_i)\nabla p\big) - div(\mathbf{K}\rho_i(p)\alpha(s_1)\nabla s_i)$$
$$+\rho_i(p)s_i f_P = \rho_i(p)s_i^I f_I, \quad i = 1,2, \tag{2.1}$$

together with the boundary conditions (1.7) and the initial conditions (1.8, 1.9) in the Hilbert space $L^2(\Omega)$. Let $\mathcal{P}_N$ in $L^2(\Omega)$ be the projector on the first N eigenvectors of the operator

$$p \mapsto -div(\mathbf{K}\nabla p)$$

defined on $H^1_{\Gamma_1}(\Omega)$ for the boundary conditions (1.7). Let Z be defined by

$$Z(s) = \begin{cases} 0 & \text{for } s \le 0, \\ s & \text{for } s \in [0,1], \\ 1 & \text{for } s \ge 1. \end{cases}$$

For $N > 0$ and $\varepsilon > 0$ fixed, consider $(p^{N,\epsilon}, s_1^{N,\epsilon})$ solution to

$$\frac{(\phi\rho_1)(x,p^{N,\varepsilon})Z(s_1^{N,\varepsilon}) - \phi^*\rho_1^* s_1^*}{h}$$
$$-\mathcal{P}_N^* div\Big(\mathbf{K}\big(\frac{\rho_1(x,p^{N,\varepsilon})}{(\phi\rho_2)(x,p^{N,\varepsilon})}M_1(s_1^{N,\varepsilon})\Big(\nabla\mathcal{P}_N\big(\int_0^{p^{N,\varepsilon}} \phi\rho_2(x,q)dq\big)$$
$$- \int_0^{p^{N,\varepsilon}} \nabla\phi(x,q)\,\rho_2(q)dq\Big)$$
$$-div\Big(\mathbf{K}\rho_1(p^{N,\varepsilon})\alpha(s_1^{N,\varepsilon})\nabla s_1^{N,\varepsilon})\Big) + \rho_1(p^{N,\varepsilon})Z(s_1^{N,\varepsilon})f_P$$
$$= \rho_1(p^{N,\varepsilon})s_1^I f_I, \tag{2.2}$$

$$\frac{(\phi\rho_2)(x,p^{N,\varepsilon})Z(s_2^{N,\varepsilon})-\phi^*\rho_2^*s_2^*}{h} - div\Big(\mathbf{K}\rho_2(p^{N,\varepsilon})(M_2+\varepsilon)\nabla p^{N,\varepsilon}\Big)$$
$$-div\Big(\mathbf{K}\rho_1(p^{N,\varepsilon})\alpha(s_1^{N,\varepsilon})\nabla s_2^{N,\varepsilon}\Big) + \rho_2(p^{N,\varepsilon})Z(s_2^{N,\varepsilon})f_P$$
$$= \rho_2(p^{N,\varepsilon})s_2^I f_I, \quad (2.3)$$

together with the boundary condition (1.7) and the initial conditions (1.8, 1.9). $\mathcal{P}_N^*$ is the adjoint operator of $\mathcal{P}_N$ for the scalar product of $L^2(\Omega)$. First, the existence of solutions to (2.2, 2.3) is performed in the following proposition where the dependence of solutions on parameters N and ε is omitted.

Proposition 1. *Let $\phi^*\rho_i^*s_i^* \in L^2$. Then there exits $(s_1,p) \in H_{\Gamma_1}(\Omega)\times H_{\Gamma_1}(\Omega)$, solution to (2.1) in the following weak sense*

$$\int_\Omega \frac{(\phi\rho_1)(x,p)Z(s_1)-\phi^*\rho_1^*s_1^*}{h}\varphi dx$$
$$+\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)}M_1(s_1)\mathbf{K}\Big[\nabla_x\mathcal{P}_N(\int_0^p(\phi\rho_2)(x,q)dq)$$
$$-\int_0^{p^{N,\varepsilon}}\nabla\phi(x,q)\,\rho_2(q)dq\Big]\cdot\mathcal{P}_N(\nabla\varphi)dx$$
$$+\int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_1\cdot\nabla\varphi dx + \int_{Q_T}\rho_1(p)Z(s_1)f_P\varphi dx$$
$$=\int_{Q_T}\rho_1(p)s_1^I f_I\varphi dx, \quad (2.4)$$

$$\int_\Omega \frac{(\phi\rho_2)(x,p)Z(s_2)-\phi^*\rho_2^*s_2^*}{h}\xi dx + \int_\Omega \rho_2(p)(M_2+\varepsilon)\mathbf{K}\nabla_x p\cdot\nabla\xi dx$$
$$+\int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_2\cdot\nabla\xi dx + \int_{Q_T}\rho_2(p)Z(s_2)f_P\xi dx$$
$$=\int_{Q_T}\rho_2(p)s_2^I f_I\xi dx, \quad (2.5)$$

for all $(\varphi,\xi)\in H_{\Gamma_1}(\Omega)\times H_{\Gamma_1}(\Omega)$.

Proof. (Proposition 1.) Consider s_1 solution to

$$\frac{(\phi\rho_1)(x,\overline{p})Z(\overline{s}_1)-\phi^*\rho_1^*s_1^*}{h}$$
$$-\mathcal{P}_N^* div\Bigg(\mathbf{K}(\frac{\rho_1(x,\overline{p})}{(\phi\rho_2)(x,\overline{p})})M_1(\overline{s}_1)\Big(\nabla\mathcal{P}_N(\int_0^{\overline{p}}(\phi\rho_2)(x,\overline{q})dq) - \int_0^{\overline{p}}\nabla\phi(x,q)\,\rho_2(q)dq\Big)\Bigg)$$
$$-div\big(\mathbf{K}\rho_1(\overline{p})\alpha(\overline{s}_1\nabla s_1)\big) + \rho_1(\overline{p})Z(\overline{s}_1)f_P = \rho_1(\overline{p})s_1^I f_I. \quad (2.6)$$

For $s_2 = 1 - s_1$, let p be solution to

$$\frac{(\phi\rho_2)(x,\overline{p})Z(\overline{s}_2) - \phi^*\rho_2^* s_2^*}{h} - div\big(\mathbf{K}\rho_2(\overline{p})(M_2(\overline{s}_2)+\varepsilon)\nabla p\big)$$
$$-div\big(\mathbf{K}\rho_1(\overline{p})\alpha(\overline{s}_1)\nabla s_1\big) + \rho_2(\overline{p})Z(\overline{s}_2)f_P = \rho_2(\overline{p})s_2^I f_I. \qquad (2.7)$$

The map $\mathcal{T}$ is well defined on $L^2(\Omega)$ by using succesively the Lax Milgram theorem in (2.6) and in (2.7). □

Lemma 2.1. *$\mathcal{T}$ is a continous and compact map from L^2 into itself.*

Proof. (Lemma 2.1.) Consider a sequence $(\overline{s}_{1,n}, \overline{p}_n)$ bounded in $L^2(\Omega)\times L^2(\Omega)$. The sequence $(s_{1,n}, p_n)$ satisfies

$$\int_\Omega \frac{(\phi\rho_1)(x,\overline{p}_n)Z(s_{1,n}) - \phi^*\rho_1^* s_1^*}{h}\varphi\, dx$$
$$+\int_\Omega \frac{\rho_1(x,\overline{p}_n)}{(\phi\rho_2)(x,\overline{p}_n)} M_1(s_1)\mathbf{K}\big(\nabla_x \mathcal{P}_N(\int_0^{\overline{p}_n}(\phi\rho_2)(x,q)dq)$$
$$-\int_0^{\overline{p}_n}\nabla\phi(x,q)\rho_2(q)dq\big)\cdot \mathcal{P}_N(\nabla\varphi)dx$$
$$+\int_\Omega \mathbf{K}\rho_1(\overline{p}_n)\alpha(s_{1,n})\nabla s_{1,n}\cdot\nabla\varphi dx + \int_{Q_T}\rho_1(\overline{p}_n)Z(s_1)f_P\varphi dx$$
$$= \int_{Q_T}\rho_1(\overline{p}_n)s_1^I f_I\varphi dx, \quad (2.8)$$

$$\int_\Omega \frac{(\phi\rho_2)(x,\overline{p}_n)Z(s_{2,n}) - \phi^*\rho_2^* s_2^*}{h}\xi dx + \int_\Omega \rho_2(\overline{p}_n)(M_2+\varepsilon)\mathbf{K}\nabla p_n\cdot\nabla\xi dx$$
$$+\int_\Omega \mathbf{K}\rho_1(\overline{p}_n)\alpha(s_1)\nabla s_{2,n}\cdot\nabla\xi dx + \int_{Q_T}\rho_2(\overline{p}_n)Z(s_{2,n})f_P\xi dx$$
$$= \int_{Q_T}\rho_2(\overline{p}_n)s_2^I f_I\xi dx, \quad (2.9)$$

for all $(\varphi,\xi)\in H_{\Gamma_1}(\Omega)\times H_{\Gamma_1}(\Omega)$.
So by taking $\varphi = s_{1,n}\in H_{\Gamma_1}(\Omega)$ in (2.8) and by using the assumptions $(H5)$ and $(H6)$, it holds that

$$\int_\Omega |\nabla s_{1,n}|^2 dx \le C + C\|s_{1,n}\|^2_{L^2(\Omega)} + C\|\nabla\mathcal{P}_N(\int_0^{p_n}(\phi\rho_2)(q)dq\|^2_{L^2(\Omega)},$$

where C is independent of n. So

$$\begin{aligned}\|\mathcal{P}_N\big(\phi(x,\overline{p}_n)\rho_2(\overline{p}_n)\nabla\overline{p}_n\big)\|_{L^2(\Omega)} &\leq C_N\|\int_0^{\overline{p}_n}\phi(x,q)\rho_2(q)dq\|_{L^2(\Omega)}\\ &\leq C_N\rho_M\phi_M\|\overline{p}_n\|_{L^2(\Omega)}.\end{aligned}$$

So from the Poincarre inequality, $(s_{1,n})_{n\in\mathbb{N}}$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$. By taking $\xi = p_n$ in (2.9), it holds that

$$\varepsilon\int_\Omega|\nabla p_n|^2dx \leq C(1+\|\nabla s_{1,n}\|^2_{L^2(\Omega)}).$$

By using the Poincarre inequality, $(p_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{\Gamma_1}(\Omega)$. Hence $\mathcal{T}$ is a compact map in $L^2(\Omega)\times L^2(\Omega)$. □

Lemma 2.2. *There exists $r>0$ such that if $(s_1,p)=\lambda\mathcal{T}(s_1,p)$ with $\lambda\in]0,1[$, then*

$$\|(s_1,p)\|_{L^2(\Omega)\times L^2(\Omega)} \leq r.$$

Proof. (Lemma 3.8.) Assume $(s_1,p)=\lambda\mathcal{T}(s_1,p)$. Then (s_1,p) satisfies

$$\begin{aligned}&\lambda\int_\Omega\frac{(\phi\rho_1)(x,p)Z(s_1)-\phi^*\rho_1^*s_1^*}{h}\varphi dx\\ &\qquad+\lambda\int_\Omega\frac{\rho_1(p)}{\phi\rho_2(p)}M_1(s_1)\mathbf{K}\Big(\mathcal{P}_N\big((\phi\rho_2)(p)\nabla p\big)\\ &\qquad\qquad-\int_0^p\nabla\phi(x,q)\rho_2(q)dq\Big)\cdot\nabla_x\mathcal{P}_N\varphi dx\\ &\qquad+\int_\Omega\mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_1\cdot\nabla\varphi dx+\int_{Q_T}\rho_1(p)Z(s_1)f_P\varphi dx\\ &\qquad\qquad=\lambda\int_{Q_T}\rho_1(p)s_1^If_I\varphi dx,\end{aligned}$$

$$\begin{aligned}&\lambda\int_\Omega\frac{(\phi\rho_2)(x,p)Z(s_2)-\phi^*\rho_2^*s_2^*}{h}\xi dx+\int_\Omega\rho_2(p)(M_2+\varepsilon)\mathbf{K}\nabla p\cdot\nabla\xi dx\\ &\qquad+\int_\Omega\mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_2\cdot\nabla\xi dx+\lambda\int_{Q_T}\rho_2(p)Z(s_2)f_P\xi dx\\ &\qquad\qquad=\lambda\int_{Q_T}\rho_2(p)s_2^If_I\xi dx,\end{aligned}\tag{2.10}$$

for all $(\varphi,\xi)\in H_{\Gamma_1}(\Omega)\times H_{\Gamma_1}(\Omega)$.
By setting $\varphi=g_1(x,p)=\int_0^p\phi(x,q)\rho_2(q)dq\in H^1_{\Gamma_1}(\Omega)$ and

$\xi = g_2(x,p) = \int_0^p \phi(x,q)\rho_1(q)dq \in H^1_{\Gamma_1}(\Omega)$ and by adding the two quantities, it holds that

$$\frac{\lambda}{h}\int_\Omega \Big((\phi\rho_1)(p)Z(s_1) - \phi^*\rho_1^*s_1^*)g_1(p) + ((\phi\rho_2)(p)Z(s_2) - \phi^*\rho_2^*s_2^*)g_2(p)\Big)dx$$
$$+\lambda\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(p)}M_1(s_1)\mathbf{K}\mathcal{P}_N\big((\phi\rho_2)(p)\nabla p\big)\cdot\mathcal{P}_N((\phi\rho_2)(p)\nabla p)dx$$
$$+\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(p)}M_1(s_1)\mathbf{K}\mathcal{P}_N\big(\int_0^p \phi(x,p)\rho_2(q)dq\big)\cdot\nabla_x\mathcal{P}_N\Big(\int_0^p \phi(x,p)\rho_2(q)dq\Big)dx$$
$$+\int_\Omega \mathbf{K}\alpha(s_1)\nabla s_1\cdot\Big(\int_0^p \nabla\phi(x,q)(\rho_2(q)-\rho_1(q)dq)\Big)dx$$
$$+\int_\Omega \rho_2(q)\,(M_2+\varepsilon)|\nabla p|^2dx$$
$$+\int_\Omega \rho_2(p)\,(M_2+\varepsilon)\mathbf{K}\nabla p\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_1(q)dq\Big)dx$$
$$+\int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_1\cdot\nabla\varphi dx + \int_{Q_T}(\rho_1(p)Z(s_1)g_1(p)+\rho_2(p)Z(s_2)g_2(p))f_P dx$$
$$= \lambda\int_{Q_T}\big(\rho_1(p)s_1^I f_I g_1(p) + \rho_2(p)s_2^I f_I g_2(p)\big)dx.$$

So we get the estimate

$$\varepsilon\int_\Omega |\nabla p|^2dx \le \Big|\int_\Omega \mathbf{K}\rho_1(p)\alpha_1(s_1)\,\nabla s_1\cdot\Big(\int_0^p \nabla\phi(x,q)(\rho_2(q)-\rho_1(q))dq\Big)\Big|$$
$$+\lambda\Big|\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)}M_1(s_1)\mathbf{K}\Big(\int_0^p \nabla_x\phi(x,p)\rho_2(q)dq\Big)\cdot\nabla_x\mathcal{P}_N\big(\int_0^p(\phi\rho_2)(q)dq)\big|dx$$
$$+\Big|\int_\Omega \rho_2(p)(M_2+\varepsilon)\nabla p\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big|$$
$$+c(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\phi^*\rho_1^*s_1^*\| + \|\phi^*\rho_2^*s_2^*\|). \quad (2.11)$$

Remark 3. *Denote that ε garantees that there is $k>0$ such that $(M_2+\varepsilon)\ge k$. M_2 does not satisfy such an inequality because $M_2(1)=0$.*

From Green formula, it holds that

$$\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)}M_1(s_1)\mathbf{K}\Big(\int_0^p \nabla_x\phi(x,p)\rho_2(q)dq\Big)\cdot\nabla_x\mathcal{P}_N\big(\int_0^p(\phi\rho_2)(q)dq\big)dx$$
$$= -\int_\Omega div\big(\frac{\rho_1(p)}{(\phi\rho_2)(x,p)}M_1(s_1)\mathbf{K}\big)\Big(\int_0^p \nabla_x\phi(x,p)\rho_2(q)dq\Big)\mathcal{P}_N\big(\int_0^p(\phi\rho_2)(q)dq\big)$$
$$-\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)}M_1(s_1)\mathbf{K}\,div\big(\int_0^p \nabla_x\phi(x,q)\rho_2(q)dq\big)\mathcal{P}_N\big(\int_0^p \phi(x,q)\rho_2(q)dq$$

with

$$
\begin{aligned}
&\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)} M_1(s_1)\mathbf{K}\, div\big(\int_0^p (\nabla\phi(x,q)\rho_2(q))dq\big)\mathcal{P}_N\big(\int_0^p \phi(x,q)\rho_2(q)dq \\
&\quad = \int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)} M_1(s_1)\mathbf{K}\left(\nabla_x p\cdot\nabla\phi(x,p)\rho_2(p)\right)\mathcal{P}_N\big(\int_0^p \phi(x,q)\rho_2(q)dq \\
&\quad\ \int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)} M_1(s_1)\mathbf{K}\left(\int_0^p \Delta\phi(x,q)\rho_2(q)dq\right)\mathcal{P}_N\big(\int_0^p \phi(x,q)\rho_2(q)dq.
\end{aligned}
$$

From Cauchy-Schwartz inequality and assumption $(H6)$, it holds that for any $\tau > 0$,

$$
\begin{aligned}
&\Big|\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)} M_1(s_1)\mathbf{K}\Big(\int_0^p \nabla\phi(x,p)\rho_2(q)dq\Big)\cdot\nabla\mathcal{P}_N\big(\int_0^p (\phi\rho_2)(q)dq\big)dx\Big| \\
&\qquad\qquad \le C + \tau(\|\nabla p\|^2_{L^2(\Omega)} + \|\nabla s\|^2_{L^2(\Omega)})
\end{aligned}
$$

From the Cauchy-Schwartz inequality, it holds that for any $\tau > 0$,

$$
\begin{aligned}
&\Big|\int_\Omega \mathbf{K}\rho_1(p)\alpha_1(s_1)\,\nabla s_1\cdot\Big(\int_0^p \nabla\phi(x,q)(\rho_2(q)-\rho_1(q))dq\Big)\Big| \\
&\qquad\qquad \le \tau\|\nabla s_1\|^2 + \widetilde{C}\|\int_0^p \nabla\phi(x,q)dq\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_\Omega \rho_2(p)(M_2+\varepsilon)\nabla p\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big| \\
&\qquad\qquad \le \tau\|\nabla p\|^2 + \widetilde{C}\|\int_0^p \nabla\phi(x,q)dq\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\lambda\Big|\int_\Omega \frac{\rho_1(p)}{(\phi\rho_2)(x,p)} M_1(s_1)\mathbf{K}\mathcal{P}_N\big((\phi\rho_2)(p)\nabla p\big)\cdot\mathcal{P}_N\Big(\int_0^p \nabla\phi(x,p)\rho_2(q)dq\Big)dx\Big| \\
&\qquad\qquad \le \lambda\tau\|\nabla p\|^2 + \widetilde{C}\|\int_0^p \nabla\phi(x,q)dq\|^2.
\end{aligned}
$$

Hence (2.11) leads to

$$
\begin{aligned}
\int_\Omega |\nabla p|^2 dx &\le \widetilde{C}\|\int_0^p \nabla\phi(x,q)dq\|^2 + \tau\|\nabla_x s_1\|^2 \\
&+ c(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\phi^*\rho_1^* s_1^*\| + \|\phi^*\rho_2^* s_2^*\|). \quad (2.12)
\end{aligned}
$$

Hence by taking $\xi = -s_1$ in (2.10), we get

$$\int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)|\nabla s_1|^2 dx = \lambda \int_\Omega \frac{(\phi\rho_2)(p)Z(s_2) - \phi^*\rho_2^* s_2^*}{h} s_1 dx$$
$$+ \int_\Omega \rho_2(p)(M_2+\varepsilon)\mathbf{K}\nabla p \cdot \nabla s_1 dx + \lambda \int_{Q_T} \rho_2(p)Z(s_2) f_P s_1 dx$$
$$+\lambda \int_{Q_T} \rho_2(p) s_2^I f_I s_1 dx.$$

From the Cauchy-Schwartz inequality, we get for any $\tau > 0$,

$$\int_\Omega \rho_2(p)(M_2+\varepsilon)\mathbf{K}\nabla p \cdot \nabla s_1 dx \leq \tau\|\nabla s_1\| + \widetilde{C}\|\nabla p\|.$$

So by chosing τ small enough, it holds that

$$\|\nabla s_1\|^2 \leq C + C\|\nabla p\|.$$

So by using the inequality (2.14), we get

$$\begin{aligned} \|\nabla s_1\|^2 &\leq C + \tau\|\nabla s_1\|^2 \\ &+ C(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\phi^*\rho_1^* s_1^*\| + \|\phi^*\rho_2^* s_2^*\|). \end{aligned}$$

Hence, by chosing τ small enough we obtain

$$\|\nabla s_1\|_{L^2(\Omega)} \leq C \tag{2.13}$$

with C independent of λ and the result follows from (2.14). □

Proof. (Proposition 1.) So the Leray-Schauder fixed point ([12]) theorem can be applied. This proves the existence of a solution to the system (2.4, 2.5). Then Proposition 1 is shown. □

By arguing as previously with $\lambda = 1$, we can prove as for (2.14) that

$$\|\nabla s_1\|_{L^2(\Omega)} \leq C_1$$

with C_1 independant of N. Then reasonning like for the proof of 2.14, we can prove that p satisfies

$$\begin{aligned} \int_\Omega |\nabla p|^2 dx &\leq \widetilde{C}\|\int_0^p \nabla\phi(x,q)dq\|^2 + \tau\|\nabla s_1\|^2 \\ &+ c_1(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\phi^*\rho_1^* s_1^*\| + \|\phi^*\rho_2^* s_2^*\|), \end{aligned} \tag{2.14}$$

with c_1 independant of N.
Therefore $(s_{1,N}, p_N)$ converge to (s_1, p) weakly in $H^1_{\Gamma_1}(\Omega)$, strongly in $L^2(\Omega)$ and $a.e$ in Ω. By arguing as in ([11]), we can pass to the limit with respect to the N variable in the system (2.4, 2.5). Thus the following system is obtained

$$\int_\Omega \frac{(\phi\rho_1)(x,p_\varepsilon)Z(s_{1,\varepsilon}) - \phi^*\rho_1^* s_1^*}{h}\varphi dx + \int_\Omega \rho_1(p_\varepsilon)M_1(s_1)\mathbf{K}\nabla p_\varepsilon \cdot (\nabla\varphi)dx$$
$$+ \int_\Omega \mathbf{K}\rho_1(p_\varepsilon)\alpha(s_{1,\varepsilon})\nabla s_{1,\varepsilon} \cdot \nabla\varphi dx + \int_{Q_T} \rho_1(p_\varepsilon)Z(s_{1,\varepsilon})f_P\varphi dx$$
$$= \int_{Q_T} \rho_1(p_\varepsilon)s_1^I f_I\varphi dx, \quad (2.15)$$

$$\int_\Omega \frac{(\phi\rho_2)(x,p_\varepsilon)Z(s_{2,\varepsilon}) - \phi^*\rho_2^* s_2^*}{h}\xi dx + \int_\Omega \rho_2(p_\varepsilon)(M_2+\varepsilon)\mathbf{K}\nabla_x p_\varepsilon \cdot \nabla\xi dx$$
$$+ \int_\Omega \mathbf{K}\rho_1(p_\varepsilon)\alpha(s_{1,\varepsilon})\nabla_x s_{2,\varepsilon} \cdot \nabla_x\xi dx + \int_{Q_T} \rho_2(p_\varepsilon)Z(s_{2,\varepsilon})f_P\xi dx$$
$$= \int_{Q_T} \rho_2(p_\varepsilon)s_2^I f_I\xi dx, \quad (2.16)$$

for all $(\varphi, \xi) \in H_{\Gamma_1}(\Omega) \times H_{\Gamma_1}(\Omega)$.
In order to obtain compactness on the sequences p_ε and $s_{1,\varepsilon}$, we shall use the following lemma which furnishes uniform bounds on ∇p_ε and $\nabla s_{1,\varepsilon}$ with respect to ε.

Lemma 2.3. *There are nonnegative constant c_1 and c_2 independent of ε such that*

$$\int_\Omega |\nabla p_\varepsilon|^2 dxdt \le c_1, \quad (2.17)$$

$$\int_\Omega \alpha(s_{1,\varepsilon})|\nabla s_{1,\varepsilon}|^2 dxdt \le c_2. \quad (2.18)$$

Proof. (Lemma 2.3.) By taking $\varphi = g_1(p) \in H^1_{\Gamma_1}(\Omega)$ in (2.15), $\xi = g_2(p) \in$

$H^1_{\Gamma_1}(\Omega)$ in (2.16) and by summing these quantities, it holds that

$$\frac{1}{h}\int_\Omega \Big(\big((\phi\rho_1)(x,p_\varepsilon Z(s_{1,\varepsilon}) - \phi^*\rho_1^* s_1^*\big)g_1(p_\varepsilon) + \big((\phi\rho_2)(x,p_\varepsilon)Z(s_{1,\varepsilon}) - \phi^*\rho_2^* s_2^*\big)g_2(p_\varepsilon)\Big)dx$$
$$+\int_\Omega \rho_1(p_\varepsilon)\rho_2(p_\varepsilon)\big(M_1(s_{1,\varepsilon}) + M_2(s_{2,\varepsilon}) + \varepsilon\big)\mathbf{K}\nabla p_\varepsilon \cdot \nabla p_\varepsilon dx$$
$$+\int_\Omega \Big(\rho_1(p_\varepsilon)Z(s_{1,\varepsilon})g_1(p_\varepsilon) + \rho_2(p_\varepsilon)Z(s_{2,\varepsilon})g_2(p_\varepsilon)\Big) f_P dx$$
$$= \int_\Omega \Big(\rho_1(p_\varepsilon)s_1^I g_1(p_\varepsilon) + \rho_2(p_\varepsilon)s_2^I g_2(p_\varepsilon)\Big) f_I dx.$$

By using assumption $(H2)$, we have

$$M_1(s_{1,\varepsilon}) + M_2(1 - s_{1,\varepsilon}) + \varepsilon \geq m_0.$$

So there is $C > 0$ independent of ε such that

$$\int_\Omega |\nabla p_\varepsilon|^2 dx \leq C(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\phi^*\rho_1^* s_1^*\|^2_{L^2(\Omega)} + \|\phi^*\rho_2^* s_2^*\|^2_{L^2(\Omega)}).$$

Therefore the inequality (2.17) is shown.
By taking $\varphi = -s_{1,\varepsilon}$ in (2.15), it holds that

$$\int_\Omega \mathbf{K}\rho_1(p_\varepsilon)\alpha(s_{1,\varepsilon})\nabla s_{1,\varepsilon} \cdot \nabla s_{1,\varepsilon} dx = \int_\Omega \frac{(\phi\rho_2)(p)Z(s_2) - \phi^*\rho_2^* s_2^*}{h} s_{1,\varepsilon} dx$$
$$+ \int_\Omega \rho_2(p)(M_2 + \varepsilon)\mathbf{K}\nabla p \cdot \nabla s_1 dx + \int_{Q_T} \rho_2(p)Z(s_2) f_P s_1 dx$$
$$+ \int_{Q_T} \rho_2(p) s_2^I f_I s_1 dx.$$

From assumptions $(H1)$, $(H4)$, $(H5)$ and by using Cauchy-Schwartz inequality, it holds that

$$\int_\Omega |\nabla s_{1,\varepsilon}|^2 dx \leq C + C_1\|\nabla p\|^2_{L^2(\Omega)} + C_2(\|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)}).$$

Then, by using (2.17), (2.18) follows. □

The passage to the limit with respect to ε is performed in the following proposition.

Proposition 2. *Let $s_i^* \geq 0$, $\rho_i^* \geq 0$, $\phi^* \geq 0$ such that $s_i^*\rho_i^*\phi^* \in L^2(\Omega)$. Then there is $(s_1, p) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ such that $0 \leq s_i \leq 1$ a.e. in Ω satisfying*

$$\int_\Omega \frac{(\phi\rho_1)(x,p)s_1 - \phi^*\rho_1^*s_1^*}{h}\varphi dx + \int_\Omega \rho_1(p)M_1(s_1)\mathbf{K}\nabla p\cdot(\nabla\varphi)dx + \int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)\nabla_x s_1\cdot\nabla\varphi dx + \int_{Q_T}\rho_1(p)s_1 f_P\varphi dx = \int_{Q_T}\rho_1(p)s_1^I f_I\varphi dx, \quad (2.19)$$

$$\int_\Omega \frac{(\phi\rho_2)(x,p)s_2 - \phi^*\rho_2^*s_2^*}{h}\xi dx + \int_\Omega \rho_2(p)M_2(s_1)\mathbf{K}\nabla p\cdot\nabla\xi dx + \int_\Omega \mathbf{K}\rho_1(p)\alpha(s_1)\nabla s_2\cdot\nabla\xi dx + \int_{Q_T}\rho_2(p_\varepsilon)s_2 f_P\xi dx = \int_{Q_T}\rho_2(p)s_2^I f_I\xi dx, \quad (2.20)$$

for all $(\varphi, \xi) \in H_{\Gamma_1}(\Omega) \times H_{\Gamma_1}(\Omega)$.

The proof is analogous to the proof given in ([11]). □

3 End of the proof of Theorem 1.1.

In this section, the aim is to pass to the limit when $h \to 0$ in order to get the continous problem in time. Consider $T > 0$, $N \in \mathbb{N}^*$ and $h = \frac{T}{N}$. Define the sequence $(s_{1,h}^n, p_h^n)_{n\in\mathbb{N}}$ by

$$p_h^0 = p^0, \quad s_{i,h}^0 = s_i^0 \text{ in } \Omega.$$

Let $(f_P)_h^{n+1}$ and $(f_I)_h^{n+1}$ be defined by

$$(f_P)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_P(\tau)d\tau, \quad (f_I)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_I(\tau)d\tau, \quad (s_i^I)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} s_i^I(\tau)d\tau.$$

For all $n \in [0, N-1]$, consider $(s^n_{1,h}, p^n_h) \in L^2(\Omega) \times L^2(\Omega)$, with $0 \le s^n_{1,h} \le 1$ and let $(s^{n+1}_{i,h}, p^{n+1}_h)$ be solution to the system

$$\begin{aligned}\frac{(\phi\rho_1)(x,p_h^{n+1})s_{1,h}^{n+1} - (\phi\rho_1)(p_h^n)s_{1,h}^n}{h} &- div\Big(\mathbf{K}(\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla p_h^{n+1}\Big)\\ -div(\mathbf{K}\rho_1(p_h^{n+1})\alpha(s_{1,h}^{n+1})&\nabla_x s_{1,h}^{n+1}) + \rho_1(p_h^{n+1})s_{1,h}^{n+1}(f_P)_h^{n+1}\\ &= \rho_1(p_h^{n+1})\big(s_1^I\big)_h^{n+1}\big(f_I\big)_h^{n+1}, \quad (3.1)\end{aligned}$$

$$\begin{aligned}\frac{(\phi\rho_2)(x,p_h^{n+1})s_{2,h}^{n+1} - (\phi\rho_2)(x,p_h^n)s_{2,h}^n}{h} &- div(\mathbf{K}\rho_2(p_h^{n+1})(M_2(s_{2,h}^{n+1}))\nabla p_h^{n+1})\\ -div\big(\mathbf{K}\rho_1(p_h^{n+1})\alpha(s_{1,h}^{n+1})&\nabla s_{2,h}^{n+1}\big) + \rho_2(p_h^{n+1})s_{1,h}^{n+1}(f_P)_h^{n+1}\\ &= \rho_2(p_h^{n+1})(s_2^I)_h^{n+1}(f_I)_h^{n+1}, \qquad (3.2)\end{aligned}$$

with the boundary conditions (1.7). Proposition 2 implies that the sequence is well defined.

Lemma 3.1. *There is C independent of h such that*

$$\begin{aligned}\frac{1}{h}\int_\Omega(\mathcal{H}_1(x,p_h^{n+1})s_{1,h}^{n+1} - \mathcal{H}_1(x,p_h^n)s_{1,h}^n + \mathcal{H}_2(x,p_h^{n+1})s_{2,h}^{n+1} - \mathcal{H}_2(x,p_h^n)s_{2,h}^n)dx\\ + \int_\Omega |\nabla p_h^{n+1}|^2 dx\\ \le \tau \int_\Omega |\nabla s_{1,h}^{n+1}|^2 dx + C(\|(f_P)_h^{n+1}\|^2_{L^2(\Omega)} + \|(f_I)_h^{n+1}\|^2_{L^2(\Omega)}), \qquad (3.3)\end{aligned}$$

$$\begin{aligned}\frac{1}{h}\int_\Omega(|\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1}|^2 - |(\phi\rho_1)(p_h^n)s_{1,h}^n|^2)dx + \int_\Omega |\nabla s_{1,h}^{n+1}|^2 dx\\ \le C(\|(f_P)_h^{n+1}\|^2_{L^2(\Omega)} + \|(f_I)_h^{n+1}\|^2_{L^2(\Omega)} + \|\nabla p_h^{n+1}\|^2_{L^2(\Omega)}). \qquad (3.4)\end{aligned}$$

Proof. (Lemma 3.1.) By reasonning as in ([11]), it holds that

$$\begin{aligned}\big[(\rho_1\phi)(x,p)s_1 - (\rho_1\phi)(p^*)s_1^*\big]g_1(p) + \big[(\rho_2\phi)(p)s_2 - (\rho_2\phi)(p^*)s_2^*\big]g_2(p)\\ \ge \mathcal{H}_1(p)s_1 - \mathcal{H}_1(p^*)s_1^* + \mathcal{H}_2(x,p)s_2 - \mathcal{H}_2(p^*)s_2^*. \ (3.5)\end{aligned}$$

By multiplying (3.1) with $g_1(p_h^{n+1})$, (3.2) with $g_2(p_h^{n+1})$ by adding the two obtained equations and by using (3.5), it holds that

$$
\begin{aligned}
\frac{1}{h}\int_\Omega \Big(\mathcal{H}_1(x,p_h^{n+1})s_{1,h}^{n+1} - \mathcal{H}_1(x,p_h^{n})s_{1,h}^{n} + \mathcal{H}_2(x,p_h^{n+1})s_{2,h}^{n+1} - \mathcal{H}_2(x,p_h^{n})s_{2,h}^{n}\Big)dx \\
+ \int_\Omega \rho_1(p_h^{n+1})\rho_2(p_h^{n+1})M(s_{1,h}^{n+1})\mathbf{K}\,|\nabla p_h^{n+1}|^2 dx \\
+ \int_\Omega (\rho_1(p_h^{n+1})s_{1,h}^{n+1}g_1(p_h^{n+1}) + \rho_2(p_h^{n+1})s_{2,h}^{n+1}g_2(p_h^{n+1}))(f_P)_h^{n+1} dx \\
\le \int_\Omega (\rho_1(p_h^{n})(s_1^I)_h^{n+1}g_1(p_h^{n+1}) + \rho_2(p_h^{n})(s_2^I)_h^{n+1}g_2(p_h^{n+1}))dx \\
+ \Big|\int_\Omega \mathbf{K}\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla p_h^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big| \\
+ \Big|\int_\Omega \mathbf{K}\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla s_{1,h}^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big| \\
+ \Big|\int_\Omega \mathbf{K}\rho_2(p_h^{n+1})M_2(s_{2,h}^{n+1})\nabla p_h^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_1(q)dq\Big)dx big\Big| \\
+ \Big|\int_\Omega \mathbf{K}\rho_2(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla s_{1,h}^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_1(q)dq\Big)dx\Big|.
\end{aligned}
$$

By using the Cauchy-Schwartz inequality, for any $\tau > 0$ there is a nonnegtive constant $\tilde{C}_\tau$, such that

$$
\begin{aligned}
\Big|\int_\Omega \mathbf{K}\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla p_h^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big| \\
\le \tilde{C} + \tau\|\nabla p_h^{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int_\Omega \mathbf{K}\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla s_{1,h}^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_2(q)dq\Big)dx\Big| \\
\le \tilde{C} + \tau\|\nabla s_{1,h}^{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int_\Omega \mathbf{K}\rho_2(p_h^{n+1})M_2(s_{2,h}^{n+1})\nabla p_h^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_1(q)dq\Big)dx\Big| \\
\le \tilde{C} + \tau\|\nabla p_h^{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int_\Omega \mathbf{K}\rho_2(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla s_{1,h}^{n+1}\cdot\Big(\int_0^p \nabla\phi(x,q)\rho_1(q)dq\Big)dx \\
\le \tilde{C} + \tau\|\nabla s_{1,h}^{n+1}\|^2.
\end{aligned}
$$

So (3.3) holds by chosing τ small enough. In order to get inequality (3.4), multiply (3.1) by $(\phi\rho_1)(p_h^{n+1})$ and integrate on Ω. So we get that

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \Big((\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1} - (\phi\rho_1)(p_h^n)s_{1,h}^n\Big)(\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1}dx \\
&\quad + \int_\Omega (\mathbf{K}\rho_1(p_h^{n+1})\,\alpha(s_{1,h}^{n+1})\,\nabla s_{1,h}^{n+1}\cdot\nabla((\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1})dx \\
&\quad + \int_\Omega \mathbf{K}\,\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\nabla p_h^{n+1}\cdot\nabla\big((\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1}\big)dx \\
&\quad + \int_\Omega \phi(p_h^{n+1})(\rho_1(p_h^{n+1}))^2(s_{1,h}^{n+1})^2(f_P)_h^{n+1}dx \\
&\leq \int_\Omega \phi(p_h^{n+1})(\rho_1(p_h^{n+1}))^2\, s_{1,h}^{n+1}\,(s_1^I)_h^{n+1}(f_I)_h^{n+1}dx.
\end{aligned}$$

From the relation

$$\begin{aligned}
\nabla((\phi\rho_1)(x,p_h^{n+1})) &= \nabla\phi(x,p_h^{n+1})\rho_1(p_h^{n+1}) + \partial_p\phi(x,p_h^{n+1})\,\nabla p_h^{n+1}\rho_1(p_h^{n+1}) \\
&+ \phi(x,p_h^{n+1})\rho_1'(p_h^{n+1})\nabla p_h^{n+1},
\end{aligned}$$

it holds that

$$\begin{aligned}
&\frac{1}{2h}\int_\Omega \Big(|(\phi\rho_1)(p_h^{n+1})s_{1,h}^{n+1}|^2 - |(\phi\rho_1)(p_h^n)\nabla s_{1,h}^n|^2\Big)dx \\
&\quad + \int_\Omega \mathbf{K}|\rho_1(p_h^{n+1})|\phi(p_h^{n+1})\alpha(s_{1,h}^{n+1})\nabla s_{1,h}^{n+1}\cdot\nabla s_{1,h}^{n+1}dx \\
&\leq \int_\Omega \mathbf{K}\rho_1(p_h^{n+1})(\alpha(s_{1,h}^{n+1})\frac{\partial}{\partial p}(\phi\rho_1)(p_h^{n+1}) + M_1(s_{1,h}^{n+1})(\phi\rho_1)(p_h^{n+1}))|\nabla p_h^{n+1}\cdot\nabla s_{1,h}^{n+1}|dx \\
&+ \int_\Omega \mathbf{K}\rho_1(p_h^{n+1})M_1(s_{1,h}^{n+1})\frac{\partial}{\partial p}(\phi\rho_1)(p_h^{n+1})|\nabla p_h^{n+1}|^2dx + C(\|(f_P)_h^{n+1}\|^2 + \|(f_I)_h^{n+1}\|^2).
\end{aligned}$$

By using the assumptions $(H1)$, $(H4)$, $(H5)$ the proof of Lemma 3.1 follows.

□

For any sequence $(u_h^n)_{n\in\mathbb{N}}$, denote

$$\begin{aligned}
u_h(0) &= 0, \\
u_h(t) &= \sum_{n=0}^{N-1} u_h^n\chi_{]nh,(n+1)h[}(t), \quad \forall t\in]0,T], \qquad (3.6) \\
\tilde{u}_h(t) &= \sum_{n=0}^{N-1}(1+n-\frac{t}{h})u_h^n + (\frac{t}{h})nu_h^{n+1}\chi_{[nh,(n+1)h]}(t), \quad \forall t\in[0,T]. \qquad (3.7)
\end{aligned}$$

So,

$$\partial_t \tilde{u}_h = \frac{1}{h}\sum_{n=0}^{N-1}\Big(u_h^{n+1}-u_h^n\Big)\chi_{]nh,(n+1)h[}(t), \quad \forall t\in[0,T]\backslash\{\cup_{n=0}^N nh\}.$$

Let p_h and s_h be defined as in (3.6). In the same way we define $r_{i,h}^n$ the function such that $r_{i,h}^n = (\phi\rho_i)(p_h^n)s_{1,h}$ and $\tilde{r}_{i,h}^n$ the associated function as in (3.7).

Analogously we can define the functions $f_{P,h}$ and $f_{I,h}$ associated to $f_{P,h}^{n+1}$ and $f_{I,h}^{n+1}$.

Lemma 3.2.

$$(p_h)_h \textit{ is uniformly bounded in } L^2(0,T;H_{\Gamma_1}(\Omega)), \tag{3.8}$$

$$(s_{1,h})_h \textit{ is uniformly bounded in } L^2(0,T;H_{\Gamma_1}(\Omega)), \tag{3.9}$$

$$(r_{i,h})_h \textit{ is uniformly bounded in } L^2(0,T;H_{\Gamma_1}(\Omega)),\ i=1,2, \tag{3.10}$$

$$(\partial_t\tilde{r}_{i,h})_h \textit{ is uniformly bounded in } L^2(0,T;(H_{\Gamma_1}(\Omega))'),\ i=1,2. \tag{3.11}$$

Proof. (Lemma 3.2.) Multiply by (3.3) by h and sum from $n=0$ to $n=N-1$,

$$\begin{aligned}\int_\Omega\Big(\mathcal{H}_1(p_h(T))\,s_{1,h}(T)+\mathcal{H}_2(p_h(T))\,s_{2,h}(T)\Big)dx+\int_{Q_T}|\nabla p|^2dxdt\\ \le\int_\Omega\Big(\mathcal{H}_1(p_0)\,s_{1,h}(0)+\mathcal{H}_2(p_0)\,s_{2,h}(0)\Big)dx\\ +\tau\int_{Q_T}|\nabla s_{1,h}|^2dxdt+C(\|f_P\|^2_{L^2(Q_T)}+\|f_I\|^2_{L^2(Q_T)}).\end{aligned}$$

By summing (3.4) from 0 to $n=N-1$, it holds that

$$\begin{aligned}\int_\Omega\phi\,|\rho_1(p_h(T))s_{1,h}(T)|^2dx+\int_{Q_T}|\nabla s_{1,h}|^2dxdt\\ \le\int_\Omega\phi\,|s_{1,h}(0)|^2dx+C(\|f_P\|^2_{L^2(Q_T)}+\|f_I\|^2_{L^2(Q_T)})\\ +\int_{Q_T}|\nabla_x p|^2dxdt.\end{aligned}\tag{3.12}$$

So

$$\begin{aligned}\int_{Q_T}|\nabla p|^2dxdt &\le \int_\Omega\Big(\mathcal{H}_1(p_0)\,s_{1,h}(0)+\mathcal{H}_2(p_0)\,s_{2,h}(0)\Big)dx\\ &+\ \tau\int_\Omega\phi\,|s_{1,h}(0)|^2dx+\tau\int_{Q_T}|\nabla p|^2dxdt\\ &+\ \widetilde{C}(\|f_P\|^2_{L^2(Q_T)}+\|f_I\|^2_{L^2(Q_T)}).\end{aligned}$$

And the control of ∇p follows by chosing τ small enough. The control of $\nabla s_{1,h}$ is then given by (3.12). As

$$\nabla r_{i,h} = \sum_{n=0}^{N-1} \big(\partial_p(\phi\rho_i)s_{i,h}\nabla p_h + (\phi\rho_i)(p_h)\nabla s_{i,h}\big)\chi_{]nh,(n+1)h]}.$$

$(H4)$, (3.8) and (3.9) lead to (3.10).
For any $\varphi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$,

$$\begin{aligned}\langle \partial_t \tilde{r}_{i,h}, \varphi\rangle &= -\int_{Q_T} \rho_i(p_h)M_i(s_{i,h})\mathbf{K}\nabla p_h \cdot \nabla\varphi dxdt \\ &- \int_{Q_T} \rho_i(p_h)\alpha(s_{i,h})\mathbf{K}\nabla s_{1,h} \cdot \nabla\varphi dxdt \\ &- \int_{Q_T} \rho_i(p_h)s_{i,h}f_{P,h}\varphi dxdt - \int_{Q_T} \rho_i(p_h)s_{i,h}f_{I,h}\varphi dxdt.\end{aligned}$$

So from the previous estimates, (3.11) is obtained. □

By arguing as in ([11]), we can show the following Propsition.

Lemma 3.3. *For $r_{i,h}$, $\tilde{r}_{i,h}$, $s_{1,h}$, p_h and $M_{i,h}$ defined previously, it holds that when $h \to 0$*

$$\begin{aligned}&r_{i,h} - \tilde{r}_{i,h} \to 0 \text{ strongly in } L^2(Q_T),\\ &s_{1,h} \to s_1 \text{ weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)),\\ &p_h \to p \text{ weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)),\\ &r_{i,h} \to r_i \text{ strongly in } L^2(Q_T).\end{aligned}$$

Proof. (Theorem 1.1) Consider the following weak formulation for any $i \in \{1,2\}$,

$$\begin{aligned}\langle \partial_t \tilde{r}_{i,h}, \varphi\rangle &+ \int_{Q_T} \rho_i(p_h)M_i(s_{i,h})\,\mathbf{K}\nabla_x p \cdot \nabla\varphi dxdt \\ + \int_{Q_T} \rho_i(p_h)\alpha(s_{i,h})\,\mathbf{K}\nabla s_{1,h} \cdot \nabla\varphi dxdt &+ \int_{Q_T} \rho_i(p_h)\,s_{i,h}\,f_{P,h}\varphi dxdt \\ &= \int_{Q_T} \rho_i(p_h)\,(s_i^I)_h\,f_{I,h}\varphi dxdt, \quad (3.13)\end{aligned}$$

where $\varphi \in L^2(0,T;H^1(\Omega))$.
According to Proposition 3.3, we can pass to the limit into the equation (3.13). So Theorem 1.1 is establised. □

References

[1] H.W.Alt and E. Di Benedetto. *Nonsteady flow of water and oil through inhomogeneous porous media.* Annali della Scuola Normale Superiore di Piza, XII, 3, 1985.

[2] G.Chavent and J.Jaffre. *Mathematical models and finite elements for reservoir simulation. Single phase, multiphase and multicomponent flows through porous media* Studies in Mathematics and its Applications; 17, North-Holland Publishing Comp., 1986.

[3] F.Z. Daim, R. Eymard and D.Hillhorst. *Existence of a solution for a two phase flow in porous media: the case that the porosity depends on the pressure.* Journal of Mathematical Analysis and Applications, 326(1):332-351, 2007.

[4] P.Fabrie and T.Gallouët. *Modeling welles in porous media flows*, Math. Models Methods Appl., 10 no5, 673-709, 2000.

[5] P.Fabrie and M.Langlais. *Mathematical analysis of miscible displacement in porous medium*, SIAM J.Math. Anal., 23, no 6, 1375-1392, 1992.

[6] P.Fabrie, P.Le Thiez and P.Tardy, *On a system of nonlinear elliptic and degenerate parabolic equations describing compositional water-oil flows in porous media* Nonlinear Anal., 28, no 9, 1565-1600, 1997.

[7] P.Fabrie and M.Saad. *Existence de solutions faibles pour un modèle d'écoulement triphasique en milieu poreux*, Ann.Fac.Sci. Toulouse, 2 (3) 337-373, 1993.

[8] G.Gagneux and M.Madaume-Tort *Analyse mathématique de modèles non linéaire de l'ingénérie pétrolière* Mathématiues et applications, 22, Springer, Verlag, 1995.

[9] C.Galusinski and M.Saad. *On a degenerate parabolic system for compressible immiscible two-phase flows in porous media*, Adv. in Diff Equ., 9 (11-12) 1235-1278, 2004

[10] C.Galusinski and M.Saad. *A non linear degenerate system modelling water-gas flows in porous media.*, DCDS-B, 9, 281–308, 2008.

[11] C.Galusinski and M.Saad. *Two compressible immiscible fluids in porous media.* To appear in Journ.Diff.Equ.

[12] E.Zeidler. *Nonlinear analysis and fixed-point Theorems*, Berlin, Springer-Verlag, 1993.