Dissipative hydrodynamic models for the diffusion of impurities in a gas*

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Abstract

Recently linear dissipative models of the Boltzmann equation have been introduced in [5, 7]. In this note, we consider the problem of constructing suitable hydrodynamic approximations for such models.

1 Introduction

The dissipative linear Boltzmann equation (see [5, 7] and the references therein) describes the dynamic of a set of particles with mass m interacting inelastically with a background gas in thermodynamical equilibrium composed of particles with mass $m_1 \ll m$. For example, the case of fine polluting impurities interacting with air or another gas [4].

As observed in [7], the only conserved quantity is the number of inelastic particles and as a result, a conventional hydrodynamic approach of Euler type leads to a single equation describing the advection (or advection-diffusion at the Navier-Stokes order) of inelastic particles at the velocity of the background.

The aim of this note is to find hydrodynamic models for such Boltzmann equation which contain evolution equations for the momentum and the temperature of the gas. Here, we present a closed set of dissipative Euler equations for a Maxwellian model which is a simple generalization of the one considered in [7].

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Let us mention that the problem of finding suitable hydrodynamics for inelastic interacting gases has been studied recently by several authors (see [1, 2, 3, 6] and the references therein).

The paper is organized as follows. Section 2 deals with the linear dissipative Boltzmann model and the Maxwellian approximation. Section 3 is devoted to discuss the problem of the closure of the moment equations and the derivation of a dissipative Euler system.

2 The dissipative linear Boltzmann equation

We consider the dissipative linear Boltzmann equation

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f)(t,x,v), \tag{2.1}$$

with,

$$\frac{1}{2\pi\lambda} \int_{\mathbb{R}^3_v \times S^2} B(v,w,n)[\frac{1}{e^2} f(v_*) M_1(w_*) - f(v) M_1(w)] dw dn. \tag{2.2}$$

Here, $B(v,w,n)$ denotes the collision kernel, λ the mean free path and e the restitution coefficient with $0 < e < 1$. The case $e = 1$ corresponds to the elastic collision mechanism.

For the hard spheres model, the particles are assumed to be ideally elastic balls and the corresponding collision kernel is given by

$$B(v,w,n) = |q \cdot n|, \tag{2.3}$$

with $q = v - w$. The background is assumed to be in thermodynamic equilibrium with given mean velocity u_1 and temperature T_1 i.e. its distribution function M_1 is the normalized Maxwellian given by

$$M_1(v) = \frac{1}{(2\pi R_1 T_1)^{\frac{3}{2}}} \exp\left(-\frac{(v-u_1)^2}{2R_1T_1}\right), \tag{2.4}$$

where $R_1 = K_b/m_1$ and K_b is the Boltzmann constant.

Mass ratio and inelasticity are described by the following dimensionless parameters

$$\alpha = \frac{m_1}{m_1 + m} \qquad \text{and} \qquad \beta = \frac{1-e}{2}, \tag{2.5}$$

where $0 < \alpha < 1$ and $0 < \beta < \frac{1}{2}$.

In these conditions, it's possible to prove (see [5, 7]) that the stationary equilibrium states of the collision operators are given by the Maxwellian distributions

$$M^\sharp(v) = \frac{\rho}{(2\,\pi\, RT^\sharp)^{3/2}} \exp\left(-\frac{(v-u_1)^2}{2\, RT^\sharp}\right), \qquad R = K_b/m, \tag{2.6}$$

having the same mean velocity of the background and temperature

$$T^{\sharp} = \frac{\alpha\,(1-\beta)}{1-\alpha\,(1-\beta)}\frac{R_1}{R}T_1, \tag{2.7}$$

lower than the background one.

Here by analogy with [1], we consider an approximation of the hard sphere model characterized by the assumption

$$|v-w| \simeq S(t,x), \tag{2.8}$$

where $S(t,x)$ is a suitable function which takes into account the fact that we have large relaxation rates for $|v-w|$ large and small relaxation rates for $|v-w|$ small. Clearly since v is distributed accordingly to f and w accordingly to M_1 the function S cannot be simply a function of the temperature of a single gas as in [1].

Remark 1 Since M_1 is given by (2.4) a possible choice consists in taking S as the expected value of $|v-w|$. This gives

$$S(x,t) = \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_v} |v-w| f(x,v,t) M_1(x,w)\,dw\,dv = \int_{\mathbb{R}^3_v} Z(x,v)f(v)dv, \tag{2.9}$$

with

$$Z(x,v) = \int_{\mathbb{R}^3_v} |v-w| M_1(x,w)\,dw. \tag{2.10}$$

Off course simpler choices can be done. For example, similarly to the case of a single gas, taking $S(x,t) = \mu\sqrt{T_r(x,t)}$ for a suitable constant μ, where T_r is the relative "temperature" given by

$$T_r(x,t) = \frac{1}{3\rho(x)R}\int f(x,v,t)|v-u_1(x)|^2\,dv.$$

Note that at variance with [1] here the "temperature" T_r of the inelastic gas is measured with respect to the mean velocity of the background. Thus only asymptotically for large times it will correspond to the physical temperature.

Therefore the Maxwellian model is given by

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \frac{S(t,x)}{2\pi\lambda}\int_{\mathbb{R}^3_v\times S^2} [\frac{1}{e^2}f(v_*)M_1(w_*) - f(v)M_1(w)]dwdn, \tag{2.11}$$

which corresponds to the model considered in [7] for $S(x,t) = const.$

3 Hydrodynamic limit and the Euler equation.

To avoid the term $1/e^2$ in (2.11), it is useful to consider the weak form of (2.11). More precisely, let us define with $< \cdot, \cdot >$ the inner product in $L^1(\mathbb{R}^3)$. Given any regular test-function $\varphi(v)$, we have

$$< \varphi, Q(f) >= \frac{S(t,x)}{\lambda 2\pi} \int_{\mathbb{R}^3_w} \int_{\mathbb{R}^3_v \times S^2} (\varphi(v^*) - \varphi(v)) f(v) M_1(w) dw dv dn, \tag{3.12}$$

where the post-collisional velocity v^* is defined by

$$v^* = v - 2\alpha(1-\beta)(q \cdot n)n. \tag{3.13}$$

Clearly $\varphi = 1$ is a collision invariant whereas $\varphi = v$ and $\varphi = v^2$ are not.

The existence of a Maxwellian equilibrium at non-zero temperature (2.6) allows to construct hydrodynamic models for the considered granular flow. However, here only the mass of the inelastic particles is preserved. Thus the mass ρ is the unique hydrodynamic variable and the Euler system is reduced to the single advection equation [7]

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u_1) = 0. \tag{3.14}$$

In order to perform a closure for the moment equations such that the equations for the mean velocity and the temperature of particles are kept we assume the distribution function f to be the local Maxwellian at the mean velocity and temperature of the gas

$$M(x,v,t) = \frac{\rho(x,t)}{(2\pi R T(x,t))^{\frac{3}{2}}} \exp\left(-\frac{(v-u(x,v))^2}{2RT(x,t)} \right). \tag{3.15}$$

Taking $\varphi = v$ in (3.12) leads to

$$< v, Q(M) >= \frac{-\alpha(1-\beta)S(t,x)}{\lambda \pi} \int_{\mathbb{R}^3_w} \int_{\mathbb{R}^3_v} M(v) M_1(w) (\int_{S^2} (q \cdot n) n dn) dw dv. \tag{3.16}$$

Following ([2],[5]), we get

$$\int_{S^2} (q \cdot n) n dn = \frac{4\pi}{3} q. \tag{3.17}$$

So, (3.16) has the following expression

$$< v, Q(M) >= \frac{-4\alpha(1-\beta)S(t,x)}{3\lambda} \int_{\mathbb{R}^3_w} \int_{\mathbb{R}^3_v} M(v) M_1(w) (v-w) dw dv. \tag{3.18}$$

Since

$$\int_{\mathbb{R}^3_v} vM(v)dv = \rho u \qquad \text{and} \qquad \int_{\mathbb{R}^3_v} wM_1(w)dv = u_1, \tag{3.19}$$

the first moment equation has the expression

$$\frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho T) = \frac{-4S(t,x)\alpha(1-\beta)}{3\lambda}\rho(u-u_1). \tag{3.20}$$

For the second moment, let us compute (3.12) with $\varphi = \frac{1}{2}|v|^2$. We get

$$\begin{aligned} < \frac{1}{2}|v|^2, Q(M) > &= \frac{S(t,x)}{\lambda\pi}\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v\times S^2}\big[-\alpha(1-\beta)(q\cdot n)(v\cdot n) \\ &+ \alpha^2(1-\beta)^2|q\cdot n|^2\big]M(v)M_1(w)dwdvdn. \end{aligned} \tag{3.21}$$

Reasoning as in ([2],[5]), it holds that

$$\int_{S^2}|q\cdot n|^2 dn = \frac{4\pi}{3}|q|^2, \tag{3.22}$$

$$\int_{S^2}(q\cdot n)(v\cdot n)dn = \frac{4\pi}{3}(q\cdot v). \tag{3.23}$$

So, integrating the right-hand side of (3.21) with respect to the n variable and using (3.22) leads to

$$\begin{aligned} < \frac{1}{2}|v|^2, Q(M) > &= \frac{-4\alpha(1-\beta)S(t,x)}{3\lambda}\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}(q\cdot v)M(v)M_1(w)dwdv \\ &+ \frac{4\alpha^2(1-\beta)^2S(t,x)}{3\lambda}\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}|q|^2M(v)M_1(w)dwdv. \end{aligned} \tag{3.24}$$

Since $q = v - w$ we obtain

$$\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}|q|^2M(v)M_1(w)dwdv = \int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}(|v|^2 - 2v\cdot w + |w|^2)M(v)M_1(w)dwdv. \tag{3.25}$$

From the identity

$$\frac{1}{2}\int_{\mathbb{R}^3_v}|v|^2M(v)dv = \rho(\frac{1}{2}|u|^2 + \frac{3}{2}RT),$$

it follows that

$$\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}|q|^2 M(v)M_1(w)dwdv = \rho(3RT + 3R_1T_1 + |u|^2 + |u_1|^2 - 2u_1\cdot u), \tag{3.26}$$

and

$$\int_{\mathbb{R}^3_w}\int_{\mathbb{R}^3_v}(|v|^2 - v\cdot w)M(v)M_1(w)dwdv = \rho(3RT + |u|^2 - u\cdot u_1). \tag{3.27}$$

Finally, using (3.26)-(3.27) in (3.24), we get the following dissipative Euler system

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla_x\cdot(\rho u) &= 0,\\
\frac{\partial\rho u}{\partial t} + \nabla_x\cdot(\rho u\otimes u) + \nabla_x(\rho T) &= \frac{8S(t,x)\alpha(1-\beta)}{3\lambda}\rho(u_1 - u),\\
\frac{\partial}{\partial t}\big(\rho(\frac{1}{2}|u|^2 + \frac{3}{2}T)\big) + \nabla_x\cdot\big(\rho u(\frac{1}{2}|u|^2 + \frac{5}{2}T)\big) &= \frac{4S(t,x)\alpha(1-\beta)}{3\lambda}\rho D(x,t),
\end{aligned} \tag{3.28}$$

where

$$D(x,t) = \alpha(1-\beta)(3RT + 3R_1T_1 + |u|^2 + |u_1|^2 - 2u_1\cdot u) - (3RT + |u|^2 - u\cdot u_1). \tag{3.29}$$

4 Conclusion

We derived an hydrodynamic approximation for linear dissipative Boltzmann equations for Maxwell molecules that keep the evolution equations for the mean velocity and the temperature of particles. To this aim the closure of the moment system is performed with respect to a local Maxwellian state which is not a local equilibrium state for the Boltzmann operator. In this way a dissipative Euler system is obtained. Extensions to the hard sphere case are actually under study.

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