A new approach of the Ellipsoidal Statistical Model.

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Abstract

In this paper we aim to introduce a systematic way to derive relaxation terms for the Boltzmann equation based under minimization problem of the entropy under moments constraints [7], [11]. In particular the moment constraints and corresponding coefficients are linked with the eigenfunctions and eigenvalues of the linearized collision operator through the Chapman-Enskog expansion. Then we deduce from this expansion a **single** relaxation term of BGK-type. Here we stop the moments constraints at the order 2 in the velocity v and recover the Ellipsoidal Statistical model [8].

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1 Introduction.

Solving the Boltzmann equation [6] involves computing a collisional term which is a very hard task. Different approaches can be considered: either find approximating models of the interaction term that are easier to compute or/and try to improve numerical approximations. We do not want here to give a huge review of all works that have been done in the two directions but to follow a way that was initiated by Bhatnagar, Gross and Krook [4] and their famous BGK model. Namely the BGK model consists in replacing the interaction term with the relaxation term

$$R(f) = \lambda(\mathcal{M} - f) \tag{1}$$

where $f = f(t,x,v)$ is the distribution function with $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, $\mathcal{M} = \mathcal{M}(t,x,v)$ is the local equilibrium function (Maxwellian distribution) defined by

$$\mathcal{M}(v) = \frac{1}{(2\pi T)^{3/2}} \exp\left(-\frac{(v-u)^2}{2T}\right) \tag{2}$$

with

$$\rho(t,x) = \int_{\mathbb{R}^3} f dv,\ u(t,x) = \frac{1}{\rho}\int_{\mathbb{R}^3} v f dv,\ T(t,x) = \frac{1}{3\rho}\int_{\mathbb{R}^3} |v-u|^2 f dv. \tag{3}$$

Here ρ, u, T are the local mass, density and temperature of the distribution function and $\lambda = \lambda(\rho, T)$ is a relaxation coefficient. The interest of such a model is that it inherits most important physical properties of the true one: mass, momentum and energy are conserved, the (mathematical) entropy decays and equilibrium states are the Maxwellian distributions. Unfortunately the BGK model does not allow to recover the true Navier-Stokes equation at the hydrodynamic limit. That is a correct Prandtl number Pr which defines the ratio between the viscosity μ and thermal conductivity κ coefficients

$$Pr = \frac{5}{2}\frac{\mu}{\kappa}.$$

Different models have been proposed to recover the correct Prandtl number. The most famous is the Ellipsoidal Statistical model (ES or ES-BGK model) [8]. Other models were proposed later on in [5] and finally in [12]. For numerical comparisons between the BGK model, the ES-BGK model and eventually other models one refers to [9],[1].

A new approach of relaxation models can also be found in [7]. Roughly speaking the idea is to add different relaxation terms toward "generalized" Maxwellians that are associated to different relaxation coefficients.

Finally one of the author [11] has proposed in the case of Maxwellian molecules a relaxation term of the form $\rho(G-f)$ that **fits exactly** to the Boltzmann collision operator in the following weak sense

$$\int \rho(G-f)m(v)dv = \int Q(f,f)m(v)dv \quad \forall m \in \mathbf{P} \tag{4}$$

where $\mathbf{P}$ can be any polynomial space. This model allows to recover the right Prandtl number. Unfortunately this model cannot be extended to other potentials. Here G is the solution to the minimization problem of the entropy functional $\mathcal{H}(f)$ where

$$\mathcal{H}(f) = \int_{\mathbb{R}^3} f\,(\ln f - 1)\,dv \tag{5}$$

under the moment constraints (4). Such minimization problems are the basis of the work by D. Levermore [7]).

In this paper we rather look for a single relaxation term of the form $\lambda(G-f)$ that can generalize the BGK-type models. The idea is the following: G is defined through a minimization principle of the entropy under moments constraints. More precisely let $\mathbf{P}$ be a given polynomial space of maximum order p with vector basis $\mathbf{m}(v) = (m_1(v), ..., m_N(v))^T$, λ and $(\lambda_i)_{i=1,...,N}$ be some nonnegative real number and $f \in L^1$ be a nonnegative function such that $\int f(v)\,(1+|v|^p)dv < +\infty$. Then we introduce the following minimization problem

"Find G solution to the minimization problem

$$G = \min_{g \in \mathcal{C}_f} \mathcal{H}(g) \tag{6}$$

where $\mathcal{C}_f$ is the set of functions $g \geq 0$ s.t the following equations hold

$$\int_{\mathbb{R}^3} \lambda(g-f)m_i(v)\,dv = -\lambda_i \int_{\mathbb{R}^3} f m_i(v)dv, \;\; \forall i = 1, ..., N.\text{''} \tag{7}$$

λ together with $(\lambda_i)_{i=1,...,N}$ are seek so as to satisfy the physical properties of $Q(f,f)$. In particular $(\lambda_i)_{i=1,...,N}$ are parameters to be chosen later for fitting the hydrodynamic limit of the Boltzmann equation at different level with respect to the Knudsen number (see [6]) . Notice that the idea is different from the one developed in [11] where the relaxation model is set through an approximation theory principle.

The question whether this problem admits a solution or not is subjected to the value of λ and $(\lambda_i)_{i=1,...,N}$ and to the so-called realizability moments problem (see [7],[10], [11]).

In this paper we restrict our approach to the case of

$$\mathbf{P} = \text{span}[1, v, v \otimes v].$$

It is divided as follows. After setting the problem we construct in section 3 a relaxation operator. This relaxation operator shows up to be the ES-BGK model henceforth giving a new approach of it. Section 4 is devoted to the study of the entropy dissipation of our model. We obtain a different proof of the entropy decay than that in [2]. Finally we compare in section 5 the different relaxation models -ES-BGK, the one of Bouchut and Perthame [5] and and the one of Struchtrup ([12], [9])- enjoying an entropy decay and giving the right Prandtl number. We show a general variational principle that gives new proofs for the existence of those models ([5], [12], [9]) and we end up with a discussion comparing the different models.

2 Setting of the problem.

Recall that in our paper we look for a relaxation model of the form

$$R(f) = \lambda(G - f) \tag{8}$$

where G is the solution to the following minimization problem

$$G = \min_{g \in \mathcal{C}_f} \mathcal{H}(g). \tag{9}$$

$\mathcal{C}_f$ is the set of functions $g \geq 0$ $s.t$ the following equations hold

$$\int_{\mathbb{R}^3} (1, v, |v|^2)\, g dv = \int_{\mathbb{R}^3} (1, v, |v|^2)\, f dv, \tag{10}$$

$$\int_{\mathbb{R}^3} \lambda(g - f) A(V)\, dv = -\lambda_1 \int_{\mathbb{R}^3} f A(V) dv, \tag{11}$$

where $A(v)$ are the Sonine polynomials defined by

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 Id, \tag{12}$$

Id being the matrix identity in $\mathbb{R}^3$ and

$$V = \frac{v - u}{\sqrt{T}}. \tag{13}$$

u and T are defined by (3).
λ is the relaxation coefficient to G while λ_1 is the relaxation rate of the moment

$$\int_{\mathbb{R}^3} f A(V) dv.$$

This means that we restrict our approach to the case where the polynomial space is

$$\mathbf{P} = \text{span}[1, v, v \otimes v].$$

The equations (10) just express the conservation laws of mass, momemtum and energy. The relation (11) can be interpreted in the following sense

$$\int_{\mathbb{R}^3} f A(V) dv = \int_{\mathbb{R}^3} \mathcal{L}(\frac{f}{\mathcal{M}})\, \mathcal{L}^{-1}(A(V))\, \mathcal{M}(v) dv,$$

where $\mathcal{L}(f)$ is the linearized Boltzmann operator [6]. The interest is that $\mathcal{L}^{-1}(A(V)) = \alpha(T, |V|)A(V)$ is somehow an "eigenfunctions" of the linearized collision operator. Then $R(f)$ is chosen to mimic the behaviour of $\mathcal{L}(f)$ or equivalently the behaviour of $Q(f, f)$ closed to equilibrium. This explains why λ and λ_1 are seek so as to match the hydrodynamic limit of the true Boltzmann equation.

Now introducing the stress tensor Θ,

$$\Theta = \frac{1}{\rho} \int_{\mathbb{R}^3} c \otimes c\, f dv, \quad c = v - u, \tag{14}$$

(11) can be rewritten in the form

$$\frac{1}{\rho} \int_{\mathbb{R}^3} c \otimes c\, g dv = (1 - \frac{\lambda_1}{\lambda})\Theta + \frac{\lambda_1}{\lambda} T Id. \tag{15}$$

We see that this equation only depends on the fraction $\frac{\lambda_1}{\lambda}$ which must be dimensionless.
In order to compare the present result with the other works on the Ellipsoidal Statistical Model, we set $\nu = 1 - \frac{\lambda_1}{\lambda}$ and (15) takes the form

$$\frac{1}{\rho} \int_{\mathbb{R}^3} c \otimes c\, g dv = \nu\Theta + (1 - \nu) T Id = \mathcal{T}. \tag{16}$$

3 Construction of the Ellipsoidal Statistical Model.

In this section we construct the ES-BGK model performed in three steps.

1. We solve exactly the minimization problem for different values of the ratio between λ and λ_1 relaxation coefficients.

2. By using the hydrodynamic limit of our formal model through a Chapmann-Engskog expansion up to the order 1 one can define a relaxation rate λ_1 for given moments (namely the Sonine polynomials $A(v)$ defined below) of the distribution function **independently** of the relaxation coefficient λ.
   Then the relaxation coefficient λ is defined by equating the different terms appearing in the Chapmann-Engskog expansion at order 0.

3. λ is set so as to match with the Navier-Stokes equation at the hydrodynamic limit with the true Prandtl number $Pr = \frac{2}{3}$. From this we recover the ES-BGK with a new interpretation.

3.1 Realisability conditions for the existence of the solution to the minimization problem (9).

Our first purpose is to construct the relaxation function G. As we are going to see a necessary and sufficient coefficient for the problem (9) to possess a solution is $\nu \in [-\frac{1}{2}, 1[$. This improves the result found in [2]. So let us consider the weighted space $L^1_2 = \{f \in L^1 \ s.t. (1 + |v|^2)f \in L^1\}$. Then we have the following result

Theorem 1. *For all nonnegative functions $f \in L^1_2$ and $\nu \in [-\frac{1}{2}, 1[$, the tensor $\mathcal{T}$ is symetric positive definite and the problem (9) admits a unique solution G defined by*

$$G(v) = \frac{1}{\sqrt{det(2\pi\mathcal{T})}} \exp\left(-\frac{1}{2}\langle c, \mathcal{T}^{-1}c\rangle\right). \tag{17}$$

Conversely, if (9) admits a unique solution for all nonnegative function $f \in L^1_2$, then $\nu \in [-\frac{1}{2}, 1[$.

The existence of a solution to the minimization problem (9) requires the condition $\mathcal{C}_f \neq \emptyset$. In [2] this is done by exhibiting the function G (17) for $\mathcal{T}$ semi definite positive and then proving that this function is the solution to the minimization problem. Nevertheless this can be done by using more general arguments [7], [10], [11]. As shown in those papers a necessary and sufficient condition is $\mathcal{C}_f \neq \emptyset$ for $\mathbf{P}$ of maximal degree two.

Lemma 1. *Let χ be a symetric definite tensor of order two and consider the set $\mathcal{C}$ of functions $g \geq 0$ s.t.*

$$\int g dv = \rho, \quad \frac{1}{\rho}\int v g dv = u, \quad \frac{1}{3\rho}\int |v-u|^2 g dv = T \tag{18}$$

for some (ρ, u, T) in $\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}_+^*$ and

$$\frac{1}{\rho}\int_{\mathbb{R}^3} c \otimes c\, g dv = \chi.$$

then $\mathcal{C}$ is not empty.

The proof of lemma 1 is given in appendix.

Proof. (Theorem 1)
For the sake of clarity we recall here why $\mathcal{C}_f \neq \emptyset$ (see [2]).
By denoting with $\theta_1, \theta_2, \theta_3$ the eigenvalues of Θ the eigenvalues of $\mathcal{T}$ are

$$\frac{(1-\nu)}{3}(\theta_1 + \theta_2 + \theta_3) + \nu\theta_i, \qquad i = 1, 2, 3.$$

These values are obviously nonnegative for $\nu \in [0, 1]$. The previous expression writes for $i = 1$

$$\frac{1+2\nu}{3}\theta_1 + \frac{(1-\nu)}{3}(\theta_2 + \theta_3)$$

and is therefore nonnegative for $\nu \geq -\frac{1}{2}$. So $\mathcal{T}$ is symmetric definite positive. Then $\mathcal{C}_f$ is non empty according to Lemma 1. Since the condition (16) is of power 2 in v then the moments (10, 15) are realizable ([10], [11]) and the minimization problem 9 has a unique solution which writes (17).

Conversely let f be a nonnegative function in L_2^1 and Θ be defined as in (14). Θ is symetric and one can diagonalize it as $\Theta = P^t \Delta P$ where the diagonal terms of Δ are $\theta_1, \theta_2, \theta_3$ with $0 < \theta_1, \theta_2, \theta_3 < 3T$ and $\theta_1 + \theta_2 + \theta_3 = 3T$.

Now if (9) admits a solution one can compute this solution which is exactly given by (17). Both Θ and $\mathcal{T}^{-1}$ can be diagonalized in the same basis such that G writes

$$G(v) = \frac{1}{\sqrt{det(2\pi\mathcal{T})}} \exp\left(-\frac{1}{2}\langle c, P^t\, D^{-1}\, Pc\rangle\right).$$

Clearly each term of the diagonal matrix D^{-1} or equivalently D satisfies the same conditions as Δ. That is writing (16) in the diagonalization basis

$$\theta_i \nu + (1-\nu)T \ \in]0, 3T[\qquad \forall i = 1,2,3. \tag{19}$$

Since those conditions must be valid for all f then letting for example θ_1 tend to 0 gives $\nu \leq 1$ and letting θ_1 tend to $3T$ gives $\nu \geq -\frac{1}{2}$. □

3.2 Definitions of λ_1 and λ: the Ellipsoidal Statistical model

The classical way to obtain the hydrodynamic limit of (1) consists in expanding f in power of ε around the local equilibrium function $\mathcal{M}$ (Chapman-Engskog expansion).

$$f = \mathcal{M}(1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots), \tag{20}$$

and then inserting it in (1). In our case we just have to replace the interaction term $Q(f,f)$ with $\lambda(G-f)$ and since G depends itself on f one has to expand G as

$$G = \mathcal{M}(1 + \varepsilon \frac{\partial G}{\partial \Theta} : \Theta^{(1)} + \dots).$$

Then comparing the Chapman-Engskog expansion of the kinetic equation

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x\right) f = \frac{\lambda}{\varepsilon}(G - f).$$

at order 0 gives

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x\right) \mathcal{M} = \lambda(\frac{\partial G}{\partial \Theta} : \Theta^{(1)} - f^{(1)})\, \mathcal{M}.$$

And so this provides as usual

$$\left(A(V) : D(u) + \frac{B(V)}{\sqrt{T}} \cdot \nabla_x T\right) \mathcal{M} = \lambda(\frac{\partial G}{\partial \Theta} : \Theta^{(1)} - f^{(1)})\mathcal{M}, \tag{21}$$

where the stress tensor $D(u)$ is defined as

$$D(u) = \frac{1}{2}(\nabla_x u + \nabla_x u^t) - \frac{1}{3} div(u) Id$$

and

$$B(V) = \frac{V}{2}(V^2 - \frac{5}{2}).$$

One can also expand equation (11) (replacing g with G) and compare the left and right hand sides at order 1.

$$\lambda \int_{\mathbb{R}^3} (\frac{\partial G}{\partial \Theta} : \Theta_1 - f^{(1)})\mathcal{M} dv = -\lambda_1 \int_{\mathbb{R}^3} f^{(1)} \mathcal{M} A(v) dv. \tag{22}$$

Inserting it in equation (21) gives after multiplication with $A(V)$ and integration

$$\int_{\mathbb{R}^3} (A(V) : D(u))\mathcal{M} A(V) dv = -\lambda_1 \int_{\mathbb{R}^3} A(V)\mathcal{M} f^{(1)} dv.$$

Hence

$$\int_{\mathbb{R}^3} A(V)\mathcal{M} f^{(1)} dv = -\frac{1}{\lambda_1} \int_{\mathbb{R}^3} A(V) : A(V)\mathcal{M} dv \, D(u).$$

This sets a relationship between the relaxation rate λ_1 and the viscosity μ

$$\lambda_1 = \frac{\rho T}{\mu} \tag{23}$$

which defines λ_1. Remark that this definition does not depend on λ.

Now

$$\lambda = \frac{\lambda_1}{1-\nu} = \frac{\rho T}{\mu(1-\nu)} \tag{24}$$

which entirely defines our relaxation model

$$R(f) = \frac{\rho T}{\mu(1-\nu)}(G - f) \tag{25}$$

with

$$G(v) = \frac{1}{\sqrt{det(2\pi \mathcal{T})}} \exp\left(-\frac{1}{2}\langle c, \mathcal{T}^{-1} c\rangle\right) \tag{26}$$

and $\mathcal{T}$ is defined by (16) (recall also the equality (24)). This turns out to be the Ellipsoidal Statistical model [8].

3.3 Thermal conductivity and Prandtl number.

Multiplying (21) with $B(V)$ and integrating with respect to v yields

$$\begin{aligned}\int_{\mathbb{R}^3} B(V) f^{(1)} \mathcal{M} dv &= -\frac{1}{3\lambda}\left(\int_{\mathbb{R}^3} B(V)\cdot B(V)\mathcal{M} dv\right)\frac{\nabla_x T}{\sqrt{T}} \\ &= -\frac{5}{2\lambda}\frac{\nabla_x T}{\sqrt{T}} \\ &= -\frac{\kappa}{\rho T}\frac{\nabla_x T}{\sqrt{T}}.\end{aligned}$$

where κ is the thermal conductivity of our model. This sets the other relationship between λ and κ as

$$\lambda = \frac{5}{2}\frac{\rho T}{\kappa}.$$

This means that the Prandtl number of this model is

$$Pr = \frac{5}{2}\frac{\mu}{\kappa} = \frac{\lambda}{\lambda_1} = \frac{1}{1-\nu}$$

Remark that two important cases of this model are those where
1- $\nu = -\frac{1}{2}$ which gives the important physical case $Pr = \frac{2}{3}$ that holds for monatomic gas,
2- $\nu = 0$ which gives the classical BGK model [4] whose Prandtl number is $Pr = 1$ and is not physical.

4 H-Theorem.

In this section we want to prove the entropy dissipation law for any $-\frac{1}{2} \leq \nu \leq 1$. Though this has already been done in [2] we give here another proof. We denote with G_ν the anisotropic Gaussian function defined by (26)

Theorem 2. *For all $-\frac{1}{2} \leq \nu \leq 1$ the entropy dissipation term satisfies*

$$D(f) = -\int (G_\nu - f) \ln f \, dv \leq 0$$

where G is defined in (26).
Moreover $D(f) < 0$ for $-\frac{1}{2} \leq \nu < 1$ with equality iff $f = \mathcal{M}$.

Proof. The convexity of $x \ln x - x$ yields the inequality

$$-\int (G_\nu - f) \ln f \, dv \leq \mathcal{H}(G_\nu) - \mathcal{H}(f) \leq \mathcal{H}(G_\nu) - \mathcal{H}(G_1)$$

The last inequality is a direct consequence of the minimization problem taking $\nu = 1$. The computation of $\mathcal{H}(G_\nu)$ gives

$$\mathcal{H}(G_\nu) = \rho \ln \frac{\rho}{\sqrt{2\pi det \mathcal{T}}} - \frac{5}{2}\rho.$$

This function is a strictly convex function of ν since it is the minimum of the entropy functional under linear constraints in ν. Therefore it is enough to prove that $G_{-\frac{1}{2}} \leq G_1$.
Remark that the minimum of $\mathcal{H}(G_\nu$ is the physical entropy

$$\mathcal{H}(\mathcal{M}) = \frac{3}{2}\rho \log \frac{\rho^{2/3}}{T} - C\rho$$

and is obtained in $\nu = 0$. Next

$$\mathcal{H}(G_\nu) - \mathcal{H}(G_1) = \frac{1}{2}\rho \log \frac{det\Theta}{det\mathcal{T}}$$

is of the same sign as

$$p(\nu) = det\Theta - det(\nu\Theta + (1-\nu)TId)$$

Let us recall that $\theta_1, \theta_2, \theta_3$ are the eigenvalues of the symmetric definite positive matrix Θ so that on one side $det\Theta = \prod_i \theta_i$ while on the other side $det(\Theta - \lambda Id) = -\prod_i(\lambda - \theta_i)$ implies

$$det\mathcal{T} = \prod_i (\nu\theta_i - (\nu - 1)T).$$

Then one has

$$p(\nu) = \prod_i \theta_i + \prod_i (\nu(T - \theta_i) - T)$$

Setting $\tilde{\theta}_i = \theta_i/3$ then we have

$$p(\nu) = T^3(\prod_i \tilde{\theta}_i + \prod_i (\nu(1 - \tilde{\theta}_i) - 1)).$$

In particular

$$p(-\frac{1}{2}) = T^3 \frac{3}{8}(3\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_3 - \tilde{\theta}_1\tilde{\theta}_2 - \tilde{\theta}_2\tilde{\theta}_3 - \tilde{\theta}_1\tilde{\theta}_3)$$

Now $0 < \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3 \leq 3$ and $\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3 = 3$ so that using Lagrange multipliers the maximum of $p(-\frac{1}{2})$ is 0 and is obtained for $\tilde{\theta}_1 = \tilde{\theta}_2 = \tilde{\theta}_3 = 1$. This corresponds to $G_0 = \mathcal{M}$. □

5 Comparison between different models giving the right Prandtl number.

In this section we want to compare different models of the BGK form giving the right Prandtl number and having an entropy decay. Namely we compare the ES-BGK model, the model of Bouchut and Perthame ([5]) and those of Struchtrup ([12], [9]). We first recall those models and then derive a general variational principle that can be applied to each of them. Finally we compare them from a theoretical point of view.

So let us first recall the model of Bouchut and Perthame ([5]). It writes

$$R_{BP}(f) = \eta\tilde{\lambda}(V)(\mathcal{M}_{\tilde{\lambda}} - f) \tag{27}$$

where $\eta = \eta(t,x) = \eta(\rho, T)$ and $\tilde{\lambda}(V) = \tilde{\lambda}(|V|)$ is a multiplier acting on

$$V(t,x,v) = \frac{v - u(t,x)}{[T(t,x)]^{\frac{1}{2}}})$$

Here ρ, u and T are the mass, velocity and temperature of f (3). Finally $\mathcal{M}_{\tilde{\lambda}}$ is a Gaussian distribution

$$\mathcal{M}_{\tilde{\lambda}}(v) = \rho_{\tilde{\lambda}} \frac{1}{(2\pi T_{\tilde{\lambda}})^{3/2}} \exp\left(-\frac{(v - u_{\tilde{\lambda}})^2}{2T_{\tilde{\lambda}}}\right) \tag{28}$$

whose parameters $(\rho_{\tilde{\lambda}}, u_{\tilde{\lambda}}, T_{\tilde{\lambda}})$ are implicitly defined through the relations

$$\int_{\mathbb{R}^3} \tilde{\lambda}(V)(1, v, |v|^2)\, \mathcal{M}_{\tilde{\lambda}} dv = \int_{\mathbb{R}^3} \tilde{\lambda}(V)(1, v, |v|^2)\, f dv. \tag{29}$$

$\tilde{\lambda}$ can be fitted to give the right Prandtl number.

H.Struchtrup ([12]) introduced later on the following relaxation term

$$R_S(f) = \bar{\lambda}_1(V)(\mathcal{M}_{\bar{\lambda}} - f) \tag{30}$$

where $\bar{\lambda}_1(V)$ is a velocity-dependant collision frequency of the form $\bar{\lambda}_1 = \zeta|V|^\alpha$. $\mathcal{M}_{\bar{\lambda}}$ is a Gaussian distribution (see above) whose parameters $(\rho_{\bar{\lambda}}, u_{\bar{\lambda}}, T_{\bar{\lambda}})$ are implicitly defined through the relations

$$\int_{\mathbb{R}^3} \bar{\lambda}(V)(1, v, |v|^2)\, \mathcal{M}_{\bar{\lambda}} dv = \int_{\mathbb{R}^3} \bar{\lambda}(V)(1, v, |v|^2)\, f dv. \tag{31}$$

Here ζ depends on macroscopic variables and $\alpha > 0$ can be fitted to give the right Prandtl number. Remark that f does not relax toward $\mathcal{M}_{\bar{\lambda}}$ at $v = u$ so that he choose the more appropriate form [9]

$$\bar{\lambda}_2(V) = \zeta(1 + C|V|^\alpha) = \zeta\bar{\lambda}(V) \tag{32}$$

Again constants $C, \alpha > 0$ can be fitted to give the right Prandtl number.

At this stage this last model seems to be just a particular choice of the relaxation model (30) since it suffices to take $\tilde{\lambda}(w) = \bar{\lambda}_1(w)$ or $\tilde{\lambda}(w) = \bar{\lambda}_2(w)$. But it is not since $\tilde{\lambda}(w)$ satisfies

$$\exists C_1, C_2, \qquad 0 < C_1 \leq \tilde{\lambda}(w) \leq C_2 < +\infty, \qquad \forall w \in \mathbb{R}^3. \tag{33}$$

Nevertheless there is some similarity as will be shown in the next section.

5.1 Variational principles

As was shown in section 3.1 (Theorem 1), the ES-BGK model enjoys a variational principle. That is G_{ES} is is the solution to the minimization problem

$$G_{ES} = \min_{g \in \mathcal{C}_f} \mathcal{H}(g).$$

where $\mathcal{H}(g)$ is the classical entropy functional (5) and C_f is the set of functions satisfying conditions (10), (11) for a given nonnegative function $f \in L^1_2(\mathbb{R}^3)$.

Now let $\omega(v) = \omega(|v|)$ be a function such that either

$$\exists\, C_1, C_2, \alpha > 0 \qquad s.t \qquad 0 < C_1 \leq \omega(v) \leq C_2(1 + |v|^\alpha) \qquad \forall v \in \mathbb{R}^3.$$

or

$$\omega = |v|^\alpha$$

It is clear that for fixed $u \in \mathbb{R}^3, T > 0$ there exists $C_2(u,T)$ such that

$$\tilde{\omega}(v)dv = \omega((v-u)/\sqrt{T})dv \tag{34}$$

is a measure in $L^1_{2+\alpha}$, respectively in

$$V = L^1_{2+\alpha} \cap \{f \in L^1 / \int |f|\,|v-u|^\alpha < +\infty\}.$$

Then one defines the weighted entropy functional

$$\mathcal{H}_{\tilde{\omega}}(f) = \int_{\mathbb{R}^3} f\,(\ln f - 1)\ \tilde{\omega}(v)dv. \tag{35}$$

One has the following general theorem.

Theorem 3. *Let $u \in \mathbb{R}^3$, $T > 0$ and $h \neq 0$ be a nonnegative function in $L^1_{2+\alpha}$ (respectively in V) then there exists a unique solution $\mathcal{M}_{\tilde{\omega}}$ to the minimization problem*

$$\mathcal{M}_{\tilde{\omega}} = \min_{g \in \mathcal{C}_h} \mathcal{H}_{\tilde{\omega}}(g). \tag{36}$$

where $\mathcal{C}_h$ is the set of function $g \geq 0$ s.t. the following relations hold

$$\int_{\mathbb{R}^3} \omega(\frac{v-u}{T^{\frac{1}{2}}})(1, v, |v|^2)\, gdv = \int_{\mathbb{R}^3} \omega(\frac{v-u}{\sqrt{T}})(1, v, |v|^2)\, hdv. \tag{37}$$

Moreover this solution is the Gaussian distribution

$$\mathcal{M}_{\tilde{\omega}}(v) = \frac{\rho_{\tilde{\omega}}}{(2\pi T_{\tilde{\omega}})^{3/2}} \exp\left(-\frac{(v-u_{\tilde{\omega}})^2}{2T_{\tilde{\omega}}}\right) \tag{38}$$

where $(\rho_{\tilde{\omega}}, u_{\tilde{\omega}}, T_{\tilde{\omega}})$ are uniquely determined by the relations (37) (replacing g with $\mathcal{M}_{\tilde{\omega}}(v)$).

Proof. One can easely extend the proof of Theorem 1 and Remark 1 [11] to the case of the weighted entropy $\mathcal{H}_{\tilde{\omega}}$. □

Now let $f \in L^1_{2+\alpha}$ (respectively in V) such that

$$\rho = \int_{\mathbb{R}^3} f dv, \; u = \frac{1}{\rho}\int_{\mathbb{R}^3} v f dv, \; T = \frac{1}{3\rho}\int_{\mathbb{R}^3} |v-u|^2 f dv, \tag{39}$$

then applying the above theorem with $h = f$ and taking successively $\omega = \tilde{\lambda}$, $\omega = \bar{\lambda}_1$ and $\omega = \bar{\lambda}_2$ proves at the same time the existence of $\mathcal{M}_{\tilde{\lambda}}(v)$ satisfying successively (29), (31) and the variational principle (36). Remark that this simplifies the proof in [5] (see Proposition 2 and Theorem 3) for the model R_{BP} (30) since the above variational principle proves at the same time the existence and uniqueness of $\mathcal{M}_{\tilde{\lambda}}$ for f satisfying (39).

5.2 Discussion

This subsection is devoted to the comparison between the ES-BGK and the two others models introduced on one hand in [5] and on the other hand in [9], [12].

Let us denote that all the models are solutions to a suitable variational problem. This allows to get the conservation laws and to obtain an H-theorem in the case of [5], [12], [9]. Nevertheless this variational principle -and especially the constraints (37) cannot be used to obtain an H-theorem for the Ellipsoidal Statistical model because of the additional condition (11)(see Theorem 2).

Considering the way to obtain the right Prandtl number one can distinguish two strategies:

- In the ES-BGK model one imposes a relaxation to 0 of the tensor $\int f\, A(V) dv$ with a suitable rate λ_1 (23). This leads to the right stress tensor with proper viscosity but not (at first sight) to the right thermal conductivity. Then one compensates this deficiency by modifying the relaxation rate λ to the local equilibrium state (Maxwellian function). Notice at this point two facts. One f tends in reality to an anisotropic Gaussian before this Gaussian becomes the local Maxwellian. Two it is quite remarkable that this anisotropic Gaussian is well defined up to the limit of $\nu = 1/2$ which on one side corresponds to the important physical value of $Pr = 2/3$ and on the other side still gives the H-theorem.

- For the other models the conditions the idea is rather to introduce velocity dependant relaxation coefficients $\tilde{\lambda}, \bar{\lambda}_1, \bar{\lambda}_2$. In [5] the relaxation coefficient $\tilde{\lambda}$ is bounded from above and below (essentially for mathematical reasons) and gives the right Prandtl number. But it is the coefficient $\eta(\rho, T)$ that gives the correct viscosity (and henceforth the correct thermal conductivity through Pr) -that is the one corresponding to a given interaction law between particles. This is exactly the choice that is made for the Ellipsoidal statistical model. On the other side the functions $\bar{\lambda}_1, \bar{\lambda}_2$ in [12], [9] are rather seek on a physical basis, that is to fit with a collision frequency expected to grow up as $|v| \to +\infty$. The interest is that from this collision frequency (though not realistic [9]) one deduces explicit expression for the viscosity and the thermal conductivity.

6 Conclusion and perspectives.

We have constructed a BGK-type operator that replaces the Boltzmann collision operator. This relaxation operator -known as Ellipsoidal statistical model [8]- is here presented through a problem of minimization of the entropy under moments constraints. More precisely beside the classical conservation laws one adds a relaxation constraint on the moment of the operator with respect to $A(V)$ (see (12), (11)). This leads to the correct choice of the Prandtl number.

A new proof of entropy dissipation for this model is given (theorem 2) together with a general variational problem for other models [5, 12, 9] giving the right Prandtl number.

We shall address in forthcoming papers the case of polyatomic gases and a possible generalization of this model to more moments constraints so as to fit with higher order(s) in the Chapman-Enskog expansion.

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A proof of lemma 1

For any function $h \in L_1^2$ the function g_α defined by

$$g_\alpha = \frac{1}{\mathcal{I}_\alpha} \frac{1}{(det(\chi))^{\frac{1}{2}}} h(\alpha \langle v-u, \chi^{-1}(v-u) \rangle)$$

with

$$\mathcal{I}_\alpha = \frac{1}{\rho}\int_{\mathbb{R}^3} \frac{1}{(det(\chi))^{\frac{1}{2}}} h(\alpha\langle v-u, \chi^{-1}(v-u)\rangle) dv$$

and

$$\alpha = \frac{2}{3}\,\frac{tr(\chi)}{T}\,\frac{\int_{\mathbb{R}^3} |c_1|^2\, h(|c|^2)dc}{\int_{\mathbb{R}^3} h(|c|^2)dc}.$$

belongs to $\mathcal{C}_f$. Indeed the first and the second constraints of (18) on the moments being satisfied let us check that the third relation relation is also satisfied.

$$\begin{aligned}\int_{\mathbb{R}^3} |v|^2 g_\alpha dv &= \frac{1}{\mathcal{I}_\alpha}\int_{\mathbb{R}^3} \frac{1}{(det(\chi))^{\frac{1}{2}}} |v-u|^2\, h(\alpha\langle v-u, \chi^{-1}(v-u)\rangle) dv \\ &+ \frac{1}{\mathcal{I}_\alpha}\int_{\mathbb{R}^3} \frac{1}{(det(\chi))^{\frac{1}{2}}} |u|^2\, h(\alpha\langle v-u, \chi^{-1}(v-u)\rangle) dv.\end{aligned}$$

χ being symetric definite and positive such that $\chi = S^2$. Then,

$$\int_{\mathbb{R}^3} |v|^2 g_\alpha dv = \frac{1}{\mathcal{I}_\alpha}\int_{\mathbb{R}^3} |S\,c|^2\, h(\alpha|c|^2)dc + \rho|u|^2.$$

But, as

$$\begin{aligned}\int_{\mathbb{R}^3} |S\,c|^2\, h(\alpha|c|^2)dc = \frac{tr(\chi)}{\alpha^{\frac{5}{2}}}\int_{\mathbb{R}^3} |c_1|^2\, h(|c|^2)dc,\\ \mathcal{I}_\alpha = \frac{1}{\rho\,\alpha^{\frac{3}{2}}}\int_{\mathbb{R}^3} h(|c|^2)dc.\end{aligned}$$

By choosing α such that

$$\alpha = \frac{2}{3}\,\frac{tr(\chi)}{T}\,\frac{\int_{\mathbb{R}^3} |c_1|^2\, h(|c|^2)dc}{\int_{\mathbb{R}^3} h(|c|^2)dc}.$$

the third condition of (18) is satisfied.