The Stationary Boltzmann equation for a two component gas in the slab.

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Abstract

The stationary Boltzmann equation for hard forces in the context of a two component gas is considered in the slab. An L^1 existence theorem is proved when one component satisfies a given indata profile and the other component satisfies diffuse reflection at the boundaries. Weak L^1 compactness is extracted from the control of the entropy production term. Trace at the boundaries are also controled.

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1 Introduction.

Consider the stationary Boltzmann problem in a slab for a two component gas

$$\xi \frac{\partial}{\partial x} f_A(x,v) = Q(f_A, f_A + f_B)(x,v), \tag{1.1}$$

$$\xi \frac{\partial}{\partial x} f_B(x,v) = Q(f_B, f_A + f_B)(x,v), \tag{1.2}$$

$$x \in [-1,1], v \in \mathbb{R}^3.$$

The collision operator Q is the Boltzmann operator

$$\begin{aligned} Q(f,g)(x,v) &= \int_{\mathbb{R}^3} \int_S B(v - v_*, \omega)[f' g'_* - f g_*] d\omega dv_*, \\ &= Q^+(f,g)(x,v) - Q^-(f,g)(x,v), \end{aligned}$$

where $Q^+(f,g) - Q^-(f,g)$ is the splitting into gain and loss term,

$$f_* = f(x, v_*), \quad f^{'} = f(x, v^{'}), \quad f_*^{'} = f'(x, v_*)$$
$$v^{'} = v - (v - v_*, \omega)\omega, \quad v_*^{'} = v_* + (v - v_*, \omega)\omega.$$

For a more general introduction to the Boltzmann equation for multicomponent gases see ([6]).

The velocity component in the x-direction is denoted by ξ, and $(v - v_*, \omega)$ denotes the Euclidean inner product in $\mathbb{R}^3$. Let ω be represented by the polar angle (with polar axis along $v - v_*$) and the azimutal angle ϕ. The function $B(v - v_*, \omega)$ is the kernel of the collision operator Q taken for hard forces as $|v - v_*|^\beta b(\theta)$, with

$$0 \leq \beta < 2, \quad b \in L^1_+([0, 2\pi]), \quad b(\theta) \geq c > 0 \quad a.e.$$

The boundary condition for the A component is the given indata profile

$$f_A(-1, v) = kM_-(v), \xi > 0, \qquad f_A(1, v) = kM_+(v), \xi < 0, \tag{1.3}$$

for some positive k.

The boundary condition for the B component is of diffuse reflection type

$$f_B(-1, v) = (\int_{\xi' < 0} |\xi'| f_B(-1, v') dv') M_-(v), \quad \xi > 0, \tag{1.4}$$
$$f_B(1, v) = (\int_{\xi' > 0} \xi' f_B(1, v') dv') M_+(v), \quad \xi < 0.$$

M_+ and M_- are given normalized Maxwellians

$$M_-(v) = \frac{1}{2\pi T_-^2} e^{-\frac{|v|^2}{2T_-}} \quad \text{and} \quad M_+(v) = \frac{1}{2\pi T_+^2} e^{-\frac{|v|^2}{2T_+}}.$$

Denote the collision frequency by

$$\nu(x, v) = \int_{\mathbb{R}^3_{v_*} \times S} B(v - v_*, \omega) f(x, v_*) dv_* d\omega.$$

In the case of two component gases, for the BGK equation, some results are obtained in ([9],[10]) from a numerical point of view, when $k = 1$. The physical context is described in those papers. f_A represents the density of a

vapor and f_B the density of a noncondensable gas. The case of multicomponent gases for the Boltzmann operator in the slab is investigated in ([11], [12], [13]), when the noncondensable gas becomes negligible. It is proved that the noncondensable gas accumulates in a thin Knudsen layer at the boundary.
In this paper, weak solutions (f_A, f_B) to the stationary problem in the sense of Definition 1.1 will be considered.

Definition 1.1. *Let M_A and M_B be given nonnegative real numbers. (f_A, f_B) is a weak solution to the stationary Boltzmann problem with the β-norms M_A and M_B, if f_A and $f_B \in L^1_{loc}((-1,1)\times\mathbb{R}^3)$, $\nu \in L^1_{loc}((-1,1)\times\mathbb{R}^3)$, $\int(1+|v|)^\beta f_A(x,v)dxdv = M_A$, $\int(1+|v|)^\beta f_B(x,v)dxdv = M_B$, and there is a constant $k>0$ such that for every test function $\varphi \in C^1_c([-1,1]\times\mathbb{R}^3)$ such that φ vanishes in a neiborhood of $\xi = 0$, and on $\{(-1,v);\xi<0\}\cup\{(1,v);\xi>0\}$,*

$$\int_{-1}^{1}\int_{\mathbb{R}^3}(\xi f_A\frac{\partial\varphi}{\partial x}+Q(f_A,f_A+f_B)\varphi)(x,v)dxdv$$

$$= k\int_{\mathbb{R}^3,\xi<0}\xi M_+(v)\varphi(1,v)dv - k\int_{\mathbb{R}^3,\xi>0}\xi M_-(v)\varphi(-1,v)dv,$$

$$\int_{-1}^{1}\int_{\mathbb{R}^3}(\xi f_B\frac{\partial\varphi}{\partial x}+Q(f_B,f_A+f_B)\varphi)(x,v)dxdv,$$

$$=\int_{\xi'<0}|\xi|M_+(v)\varphi(1,v)dv(\int_{\xi'>0}\xi' f_B(1,v')dv')$$

$$-\int_{\xi'>0}\xi M_-(v)\varphi(-1,v)dv(\int_{\xi'<0}\xi' f_B(-1,v')dv').$$

The main result of this paper is the following

Theorem 1.1. *Given β with $0\le\beta<2$, there is a weak solution to the stationary problem with β-norms equal to one.*

In the case of one component, an analogous theorem with boundary conditions of type (1.3) is proved in [1]. In the case of boundary conditions of type (1.4), an analogous theorem is also shown in [2]. The case of the Povzner equation for a one component gas with diffuse-reflection boundary in the case of hard and soft forces is investigated in ([14]).
For diffuse-reflection boundary conditions in n dimensions, the biting lemma is used in [8] to obtain (1.4). In the simpler one-dimensional frame of this paper, (1.4) is obtained by a weak L^1compactness argument from a control of the entropy outflow.

The second section of this paper is devoted to a construction of approximate solutions to the problem with a modified non symmetric collision operator. The proofs are performed by monotonicity arguments which give the uniqueness of the successive approximate solutions. In the third section, the symmetry of the collision operator is re-introduced. The weak compactness in L^1 is used in this step by noticing that the sum of the two components satisfies the stationary Boltzmann equation in the slab. Section 3 performs the passage to the limit in the traces. In section 4, some extensions of Theorem 1.1 are made, in particular, the case where M_A and M_B have any positive values.

2 Approximations with fixed total masses

Let $r > 0, m \in \mathbb{N}^*, \mu > 0, \alpha > 0, j \in \mathbb{N}^*$. Here $\chi^{r,m}$ is a C_0^∞ function with range $[0,1]$ invariant under the collision transformation J, where

$$J(v, v_*, \omega) = (v', v'_*, -\omega),$$

with $\chi^{r,m}$ also invariant under the exchange of v and v_* and such that

$$\chi^{r,m}(v, v_*, \omega) = 1 \quad , \quad min(|\xi|, |\xi_*|, |\xi'|, |\xi'_*|) \geq r,$$
$$\text{and } \chi^{r,m}(v, v_*, \omega) = 0 \quad , \quad max(|\xi|, |\xi_*|, |\xi'|, |\xi'_*|) \leq r - \frac{1}{m}.$$

The modified collision kernel $B_{m,n,\mu}$ is a positive C^∞ function approximating $min(B, \mu)$, when

$$v^2 + v_*^2 < \frac{\sqrt{n}}{2}, \text{and } |\frac{v - v_*}{|v - v_*|} \cdot \omega| > \frac{1}{m}, \text{ and } |\frac{v - v_*}{|v - v_*|} \cdot \omega| < 1 - \frac{1}{m}$$

$B_{m,n,\mu}(v, v_*, \omega) = 0$, if

$$v^2 + v_*^2 > \sqrt{n} \text{ or } |\frac{v - v_*}{|v - v_*|} \cdot \omega| < \frac{1}{2m}, \text{ or } |\frac{v - v_*}{|v - v_*|} \cdot \omega| > 1 - \frac{1}{2m}.$$

The functions φ_l are mollifiers in the x-variable defined by $\varphi_l(x) := l\varphi(lx)$, where

$$\varphi \in C_0^\infty(\mathbb{R}_v^3), \quad support(\varphi) \subset (-1, 1), \quad \varphi \geq 0, \quad \int_{-1}^{1} \varphi(x) dx = 1.$$

In order to use a fixed point theorem, consider the closed and convex subset of $L_+^1([-1, 1] \times \mathbb{R}_v^3)$,

$$K = \{f \in L_+^1([-1, 1] \times \mathbb{R}_v^3), \quad \int_{[-1,1] \times \mathbb{R}_v^3} min(\mu, (1 + |v|)^\beta) f(x, v) dx dv = 2\}.$$

A non negative function $f \in K$ and $\theta \in [0,1]$ being given, let us construct the solutions F_A and F_B of the boundary value problem

$$\alpha F_A + \xi \frac{\partial}{\partial x} F_A = \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{F_A}{1+\frac{F_A+F_B}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$
$$-F_A \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v, \quad (2.1)$$
$$F_A(-1,v) = \lambda M_-(v), \quad \xi > 0, \quad F_A(1,v) = \lambda M_+(v) \quad , \quad \xi < 0,$$

and

$$\alpha F_B + \xi \frac{\partial}{\partial x} F_B = \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{F_B}{1+\frac{F_A+F_B}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$
$$-F_B \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v, \quad (2.2)$$
$$F_B(-1,v) = \theta \lambda M_-(v), \quad \xi > 0, \quad F_B(1,v) = (1-\theta) \lambda M_+(v), \quad \xi < 0.$$

The number $\lambda > 0$ will be fixed in (2.13). Let us prove that the solution to (2.1, 2.2) is the monotone L^1-limit of the sequences $(F_A^l)_{l \in \mathbb{N}}$ and $(F_B^l)_{l \in \mathbb{N}}$ defined by

$$F_A^0 = 0,$$
$$\alpha F_A^{l+1} + \xi \frac{\partial}{\partial x} F_A^{l+1} = \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{F_A^l}{1+\frac{F_A^l+F_B^l}{j}}(y,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$
$$-F_A^{l+1} \int_{\mathbb{R}^3_{v_*} \times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in [-1,1] \times \mathbb{R}^3_{v_*}, \quad (2.3)$$
$$F_A^l(-1,v) = \lambda M_-(v), \quad \xi > 0, \quad F_A^l(1,v) = \lambda M_+(v), \quad \xi < 0,$$

and

$$F_B^0 = 0,$$
$$\alpha F_B^{l+1} + \xi \frac{\partial}{\partial x} F_B^{l+1} = \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{F_B^l}{1+\frac{F_A^l+F_B^l}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$
$$-F_B^{l+1} \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f \star \varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in [-1,1] \times \mathbb{R}^3_v, \quad (2.4)$$
$$F_B^{l+1}(-1,v) = \theta_l \lambda M_-(v), \quad \xi > 0, \quad F_B^{l+1}(1,v) = (1-\theta_l) \lambda M_+(v), \quad \xi < 0.$$

The sequences $(F^l_A)_{l\in\mathbb{N}}$ and $(F^l_B)_{l\in\mathbb{N}}$ are well defined, since they solve linear equations. They are nonnegative , for $l\geq 1$. Let $F^l=F^l_A+F^l_B$. It satisfies

$$F^0=0,$$

$$\alpha F^{l+1}+\xi\frac{\partial}{\partial x}F^{l+1}=\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{F^l}{1+\frac{F^l}{j}}(x,v')\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v^{'}_*)dv_*d\omega$$

$$-F^{l+1}\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*)dvd\omega,\quad (x,v)\in[-1,1]\times\mathbb{R}^3_v,\quad (2.5)$$

$$F^{l+1}(-1,v)=(\theta_l+1)\lambda M_-(v),\xi>0,\quad F^{l+1}(1,v)=(2-\theta_l)\lambda M_+(v),\xi<0.$$

First $F^1\geq 0$. Write the equation (2.5) in the exponential form,

$$F^{l+1}=(1+\theta_l)\lambda M_-(v)e^{-\alpha\frac{1+x}{\xi}-\int_{-\frac{1+x}{\xi}}^0\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}$$

$$+\int_{-\frac{1+x}{\xi}}^0 e^{-\alpha s-\int_s^0\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}\quad (2.6)$$

$$\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{F^l}{1+\frac{F^l}{j}}(x+s\xi,v')\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+s\xi,v'_*)dv_*dvd\omega ds,\quad \xi>0.$$

And so,

$$F^{l+1}(x,v)-F^l(x,v)=\int_{-\frac{1+x}{\xi}}^0 e^{-\alpha s-\int_s^0\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}$$

$$\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\Big(\frac{F^l}{1+\frac{F^l}{j}}(x+s\xi,v')-\frac{F^{l-1}}{1+\frac{F^{l-1}}{j}}(x+s\xi,v')\Big)\quad (2.7)$$

$$\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+s\xi,v'_*)dv_*d\omega ds,\quad \xi>0.$$

It follows that $(F^l)_{l\in\mathbb{N}}$ is a nondecreasing sequence for $\xi>0$. Moreover, $B_{m,n,\mu}$ being compactly supported because of the truncation for $v^2+v_*^2>\sqrt{n}$, $(F^l)_{l\in\mathbb{N}}$ is a bounded sequence. Hence, $(F^l)_{l\in\mathbb{N}}$ converges $a.e$ to some F in a nondecreasing way. Passing to the limit in the equation (2.7) when

$l \to +\infty$,

$$F(x,v) = (1+\theta)\lambda M_-(v) e^{-\alpha\frac{1+x}{\xi} - \int_{-\frac{1+x}{\xi}}^{0} \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau}$$
$$+ \int_{-\frac{1+x}{\xi}}^{0} e^{-\alpha s - \int_s^0 \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau} \tag{2.8}$$
$$\int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{F}{1+\frac{F}{j}}(x+s\xi, v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+s\xi, v'_*) dv_* dv d\omega ds, \quad \xi > 0.$$

For the same reasons,

$$F(x,v) = (2-\theta)\lambda M_+(v) e^{-\alpha\frac{1-x}{\xi} - \int_{\frac{1-x}{\xi}}^{0} \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau}$$
$$+ \int_{\frac{1-x}{\xi}}^{0} e^{-\alpha s - \int_s^0 \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau} \tag{2.9}$$
$$\int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{F}{1+\frac{F}{j}}(x+s\xi, v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+s\xi, v'_*) dv_* dv d\omega ds, \quad \xi < 0.$$

Let us show that F^l_A is a converging sequence. Consider Q^{A+}_l defined by

$$Q^{A+}_l = \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{F}{1+\frac{F}{j}}(x+s\xi, v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+s\xi, v'_*) dv_* dv d\omega.$$

By using that $F^A \leq F$ and a convolution with respect to the v variable ([2], [7]), it holds that Q^{A+}_l is strongly compact in L^1. So, $(F^l_A)_{l\in\mathbb{N}}$ converges to some F_A in L^1. For the same reasons, $(F^l_B)_{l\in\mathbb{N}}$ converges to some F_B. Passing to the limit in (**??**), we find that there are F_A and F_B solutions to (2.1) and (2.2).

Lemma 2.1. *The equations (2.1) and (2.2) each has a unique solution which is strictly positive.*

Proof of Lemma 2.1. Let F_A and G_A be two solutions to the equation (2.1). Consider ψ_ε, the approximation of the sign function, and ϕ_ε defined by $\phi_\varepsilon(x) = \sqrt{\varepsilon + x^2}$ a primitive of ψ_ε. Subtract the equation satisfied by G_A to the equation satisfied by F_A, multiply by $\psi_\varepsilon(F_A(x,v) - G_A(x,v))$ and integrate on $[-1,1]$,

$$\begin{aligned}
&\int_{-1}^{1}\alpha[F_A-G_A](y,v)\psi_\varepsilon(y,v)dy+\phi_\varepsilon(F_A(1,v)-G_A(1,v))\\
\le&\int_{-1}^{1}\int_{\mathbb{R}^3_{v_\star}\times S}\chi^{r,m}B_{m,n,\mu}\frac{|F_A-G_A|}{(1+\frac{F_A+F_B}{j})(1+\frac{G_A+G_B}{j})}(y,v')\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(y,v_*^{'})dv_*d\omega dy\\
&-\int_{-1}^{1}\psi_\varepsilon[F_A(y,v)-G_A(y,v)]\int_{\mathbb{R}^3_{v_*}\times S}\chi^{r,m}B_{m,n,\mu}\frac{f\star\varphi}{1+\frac{f*\varphi}{j}}(y,v_*)dv_*d\omega dy. \qquad (2.10)
\end{aligned}$$

Passing to the limit when ε tends to 0, in (2.10),

$$\begin{aligned}
&\int_{-1}^{1}\alpha|[F_A-G_A](y,v)|dy+\xi|[F_A-G_A](1,v)|\\
\le&\int_{-1}^{1}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{|F_A-G_A|}{(1+\frac{F_A+F_B}{j})(1+\frac{G_A+G_B}{j})}(y,v')\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(y,v_*^{'})dv_*d\omega dy\\
&-\int_{-1}^{1}|[F_A-G_A](y,v)|\int_{\mathbb{R}^3_{v_*}\times\mathbb{R}^3_v\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(y,v_*)dv_*d\omega dy. \qquad (2.11)
\end{aligned}$$

Integrating (2.11) on $\mathbb{R}^3_v$,

$$\begin{aligned}
&\int_{\mathbb{R}^3_v}\int_{-1}^{1}\alpha|[F_A-G_A](y,v)|dydv+\int_{\mathbb{R}^3_v}\xi|[F_A-G_A](1,v)|dv\\
\le&\int_{-1}^{1}\int_{\mathbb{R}^3_v\times\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{|F_A-G_A|}{(1+\frac{F_A+F_B}{j})(1+\frac{G_A+G_B}{j})}(y,v')\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(y,v_*^{'})dv_*d\omega dydv\\
&-\int_{-1}^{1}|[F_A-G_A](y,v)|\int_{\mathbb{R}^3_{v_*}\times\mathbb{R}^3_v\times S}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi}{1+\frac{f*\varphi}{j}}(y,v_*)dv_*d\omega dydv. \qquad (2.12)
\end{aligned}$$

As,

$$|[F_A-G_A](y,v)|\ge\frac{|F_A-G_A|}{(1+\frac{F_A+F_B}{j})(1+\frac{G_A+G_B}{j})}(y,v),$$

the right-hand side of the equation (2.12) is negative. Then, for $\xi>0$,

$$\int_{-1}^{1}\int_{\mathbb{R}^3_v}\alpha|[F_A-G_A](y,v)|dydvdydv\le 0.$$

Analogously, the same result holds for $\xi<0$. So, $F_A=G_A$. Similarly, the equation (2.2) has a unique solution. This proves Lemma 2.1. □

Let

$$f_A = \frac{F_A}{\int min(\mu, (1+|v|)^\beta) F_A(x,v) dx dv},$$
$$f_B = \frac{F_B}{\int min(\mu, (1+|v|)^\beta) F_B(x,v) dx dv}.$$

The functions f_A and f_B are well defined since F_A and F_B strictly positive. Indeed, writing the equations (2.1) and (2.2) in the exponential form,

$$F_A(x,v) \geq \lambda M_-(v) e^{-\alpha \frac{1+x}{\xi} - \int_{-\frac{1+x}{\xi}}^0 \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau}, \quad \xi > 0,$$
$$F_A(x,v) \geq \lambda M_+(v) e^{-\alpha \frac{1-x}{\xi} - \int_{\frac{1-x}{\xi}}^0 \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x+\tau\xi, v_*) dv_* d\omega d\tau}, \quad \xi < 0.$$

So using that $\int_{-1}^1 (\alpha + \nu(x,v)) dx \leq 2 + 2\mu$,

$$F_A(x,v) \geq \lambda M_-(v) e^{-\frac{2+2\mu}{\xi}}, \quad \xi > 0,$$
$$F_A(x,v) \geq \lambda M_+(v) e^{-\frac{2+2\mu}{|\xi|}}, \quad \xi < 0.$$

Analogously,

$$F_B(x,v) \geq \theta \lambda M_-(v) e^{-\frac{2+2\mu}{\xi}}, \quad \xi > 0,$$
$$F_B(x,v) \geq (1-\theta) \lambda M_+(v) e^{-\frac{2+2\mu}{|\xi|}}, \quad \xi < 0.$$

Let λ be defined by

$$\lambda = min\Big(\frac{1}{\int_{\xi>0} M_-(v) min(\mu, (1+|v|)^\beta) e^{-\frac{2+2\mu}{\xi}} dv};$$
$$\frac{1}{\int_{\xi<0} M_+(v) min(\mu, (1+|v|)^\beta) e^{-\frac{2+2\mu}{|\xi|}} dv}\Big).$$

And so,

$$\int min(\mu, (1+|v|)^\beta) F_A(x,v) dx dv \geq 1$$

and

$$\int min(\mu, (1+|v|)^{\beta}) F_B(x,v) dx dv \geq 1.$$

The functions f_A and f_B are solutions to

$$\alpha f_A + \xi \frac{\partial}{\partial x} f_A = \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f_A}{1+\frac{F}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$

$$-f_A \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v, \tag{2.13}$$

$$f_A(-1,v) = \frac{\lambda}{\int min(\mu,(1+|v|)^{\beta}) F_A(x,v) dx dv} M_-(v) \quad , \quad \xi > 0,$$

$$f_A(1,v) = \frac{\lambda}{\int min(\mu,(1+|v|)^{\beta}) F_A(x,v) dx dv} M_+(v) \quad , \quad \xi < 0,$$

and

$$\alpha f_B + \xi \frac{\partial}{\partial x} f_B = \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu}(v,v_*,\omega) \frac{f_B}{1+\frac{F}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$

$$-f_B(x,v) \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v, \tag{2.14}$$

$$f_B(-1,v) = \frac{\lambda}{\int min(\mu,(1+|v|)^{\beta}) F_B(x,v) dx dv} \theta M_-(v) \quad , \quad \xi > 0,$$

$$f_B(1,v) = \frac{\lambda}{\int min(\mu,(1+|v|)^{\beta}) F_B(x,v) dx dv} (1-\theta) M_+(v) \quad , \quad \xi < 0.$$

Recall that F is solution to

$$\alpha F + \xi \frac{\partial}{\partial x} F = \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{F}{1+\frac{F}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*^{'}) dv_* d\omega$$

$$-F \int_{\mathbb{R}^3_{v_*} \times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*) dv_* d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v, \tag{2.15}$$

$$F(-1,v) = (\theta+1)\lambda M_-(v), \quad \xi > 0, \quad F(1,v) = (2-\theta)\lambda M_+(v), \quad \xi < 0.$$

A fixed point argument will now be used in order to solve (2.13, 2.14) with $f = f_A + f_B$. Define T on $K \times [0,1]$ by $T(f,\theta) = (f_A + f_B, \tilde{\theta})$ with

$$\tilde{\theta} = \frac{\int_{\xi<0} |\xi| f_B(-1,v) dv}{\int_{\xi<0} |\xi| f_B(-1,v) dv + \int_{\xi>0} \xi f_B(1,v) dv} \tag{2.16}$$

where (f_A, f_B) is solution to (2.13, 2.14).

Lemma 2.2. *T is a continuous and compact map from $K\times[0,1]$ into itself.*

Proof of Lemma 2.2. It is clear that T maps $K\times[0,1]$ into itself. Let T_A, T_B, and $\tilde{T}$ respectively map $f\in K$ into f_A, f_B, F solutions to (2.13), (2.14, (2.17). First, $\tilde{T}$ is compact in L^1. Indeed, let f^l be a bounded sequence in $L^1([-1,1]\times\mathbb{R}^3_{v_*})$. Using the following exponential form of F^l and $F^l=\tilde{T}(f^l)$,

$$F^l(x,v)=\lambda(1+\theta_l)M_-(v)e^{-\alpha\frac{1+x}{\xi}-\int_{-\frac{1+x}{\xi}}^0\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+\tau\xi,v_*)d\tau dv_*d\omega}$$
$$+\int_{-\frac{1+x}{\xi}}^0 e^{-\alpha\frac{1+x}{\xi}-\int_s^0\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+\tau\xi,v_*)d\tau dv_*d\omega}Q_l^+(x+s\xi,v)ds, \quad (2.17)$$
$$\xi>0,$$

with

$$Q_l^+(x,v)=\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{F^l}{1+\frac{F^l}{j}}(x,v')\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x,v'_*)dv_*d\omega.$$

Because of the convolution of f^l by φ in the $x-$variable,

$$\Big(\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+\tau\xi,v_*)d\tau dv_*d\omega\Big)$$

is strongly compact in L^1. Using a convolution by a mollifier in the $v-$variable([7], [2]) , the gain term Q_l^+ is strongly compact in L^1. Hence, F^l is strongly compact in L^1 and so $\tilde{T}$ is a compact map.
Let us show that $\tilde{T}$ is a continuous map. Let f^l be a converging sequence in $L^1([-1,1]\times\mathbb{R}^3_v)$. Then, f^l is a bounded sequence in L^1. By compactness of $\tilde{T}$, F^l has a subsequence still denoted F^l converging to some F. F^l satisfies

$$\alpha F^l+\xi\frac{\partial}{\partial x}F^l=\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{F^l}{1+\frac{F^l}{j}}(x,v')\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x,v'_*)dv_*d\omega$$
$$-F^l\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3_v,\quad (2.18)$$
$$F^{\phi(l)}(-1,v)=(\theta_l+1)\lambda M_-(v),\quad \xi>0,$$
$$F^{\phi(l)}(1,v)=(2-\theta_l)\lambda M_+(v),\quad \xi<0.$$

Writing (2.18) in the exponential form,

$$F^l = (1+\theta_l)\lambda M_-(v) e^{-\alpha\frac{1+x}{\xi} - \int_{-\frac{1+x}{\xi}}^0 \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}$$

$$+\int_{-\frac{1+x}{\xi}}^0 e^{-\alpha s - \int_s^0 \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau} \tag{2.19}$$

$$\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{F^l}{1+\frac{F^l}{j}}(x+s\xi,v') \frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(x+s\xi,v'_*)dv_*dvd\omega ds, \quad \xi>0.$$

Furthermore, $F^{\phi(l)} \to F$ and $f^{\phi(l)}*\varphi \to f*\varphi$ in $L^1([-1,1]\times\mathbb{R}^3_v)$. So, there is a subsequence to $F^{\phi(l)}$ still denoted by $F^{\phi(l)}$ converging *a.e.* to F. Furthermore,

$$\chi^{r,m} B_{m,n,\mu}(v,v_*,\omega) \frac{F^l}{1+\frac{F^l}{j}}(y,v') \frac{f^l*\varphi}{1+\frac{f^l*\varphi}{j}}(y,v'_*) \le j^2 \chi^{r,m} B_{m,n,\mu}(v,v_*,\omega).$$

So, by the Lebegue theorem applied to (2.19), F is solution to

$$\alpha F + \xi\frac{\partial}{\partial x}F = \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{F}{1+\frac{F}{j}}(x,v') \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v'_*)dv_*d\omega,$$

$$-F\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi}{1+\frac{f*\varphi}{j}}(x,v_*)dv_*d\omega, \quad (x,v)\in(-1,1)\times\mathbb{R}^3_v, \tag{2.20}$$

$$F(-1,v) = (\theta+1)\lambda M_-(v), \quad \xi>0, \quad F(1,v) = (2-\theta)\lambda M_+(v), \quad \xi<0.$$

Reasoning as in the proof of Lemma 2.1, (2.20) is proved to have a unique solution. Then, the whole sequence $(F^l)_{l\in\mathbb{N}}$ converges to the solution to the equation (2.20). This proves that $\tilde{T}$ is continuous. For analogous reasons, T_A and T_B are continuous and compact maps. Therefore, the first component of T, T_A+T_B, is a compact and continuous map. It is clear that the second component of T is compact. So, it remains to show its continuity. Consider f^l converging to f in L^1 and let us show that $\tilde{\theta}_l$ converges to $\tilde{\theta}$. Recall that $\tilde{\theta}$ is defined in (2.16). Thanks to the truncation in the ξ variable $(\int_{\mathbb{R}^3_v}\xi^2 f^l(x,v)dv)$ is uniformly bounded in x. And so, $Q_l(f^l_B,f^l)$ converges to $Q(f_B,f)$ in L^1. Consider a continuously differentiable function φ defined $[-1,1]$ with values in $[0,1]$ such that $\varphi(-1)=0$ and $\varphi(1)=1$. f^l_B satisfies the equation (2.14),

$$\begin{aligned}\int_{-1}^{1}\xi\frac{\partial}{\partial x}[(f_B^l(x,v)-f_B(x,v))\varphi(x)]dx &= \int_{-1}^{1}(Q(f_B^l,f^l)-Q(f_B,f))\varphi(x)dx\\ &+ \int_{-1}^{1}(f_B^l-f_B)(x,v)\xi\frac{\partial}{\partial x}\varphi(x)dx\\ &- \alpha\int_{-1}^{1}(f_B^l-f_B)(x,v)\varphi(x)dx.\end{aligned}$$

Since φ and φ' are bounded, it follows that

$$\begin{aligned}\int_{\mathbb{R}_v^3}|\xi(f_B^l(1,v)-f_B(1,v))|dv &\leq c\int_{\mathbb{R}_v^3}\int_{-1}^{1}|(Q(f_B^l,f^l)-Q(f_B,f))|dxdv\\ &+ c\int_{\mathbb{R}_v^3}\int_{-1}^{1}|(f_B^l-f_B)(x,v)|dxdv\\ &+ c\int_{\mathbb{R}_v^3}|\int_{-1}^{1}|(f_B^l-f_B)(x,v)|dxdv.\end{aligned}$$

$Q(f_B^l,f^l)$ and f_B^l converge respectively to $Q(f_B,f)$ and f_B in $L^1([-1,1]\times\mathbb{R}_v^3)$. So, $f_B^l(1,.)$ tends to $f_B(1,.)$ in L^1. Analogously, $f_B^l(-1,.)$ tends to $f_B(-1,.)$ in L^1. Then, $\int_{\xi>0}\xi f_B^l(1,v)dv$ tends to $\int_{\xi>0}\xi f_B(1,v)dv$. In the same way, $\int_{\xi<0}|\xi|f_B^l(-1,v)dv$ tends to $\int_{\xi<0}|\xi|f_B(-1,v)dv$. Hence, the sequence $(\tilde{\theta}_l)$ converges to $\tilde{\theta}$. Finally, T is a compact and continuous map from the convex closed set $K\times[0,1]$ into itself. $\Box$
It follows from the Schauder fixed point theorem that there is (f,θ) with

$$f=f_A+f_B$$

$$\theta=\frac{\int_{\xi<0}|\xi|f_B(-1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv+\int_{\xi<0}|\xi|f_B(-1,v)dv}$$

that satisfy

$$\begin{aligned}&\alpha f_A+\xi\frac{\partial}{\partial x}f_A=\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f_A}{1+\frac{F}{j}}(x,v')\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*')dv_*d\omega\\ &\quad -f_A\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}_v^3, \quad (2.21)\\ &\qquad f_A(-1,v)=k_AM_-(v),\quad \xi>0,\quad f_A(1,v)=k_AM_+(v),\quad \xi<0\end{aligned}$$

with

$$k_A = \frac{\lambda}{\int min(\mu, (1+|v|)^{\beta}) F_A(x,v)dxdv}$$

and

$$\alpha f_B + \xi \frac{\partial}{\partial x} f_B = \int_{\mathbb{R}^3_{v_*}\times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f_B}{1+\frac{F}{j}}(x,v') \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*^{'})dv_* d\omega$$

$$-f_B \int_{\mathbb{R}^3_{v_*}\times \mathbb{S}^2} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*)dv_* d\omega, \quad (x,v)\in(-1,1)\times\mathbb{R}^3_v,$$

$$f_B(-1,v) = \lambda'\Big(\frac{\int_{\xi<0}|\xi| f_B(-1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv + \int_{\xi<0}|\xi| f_B(-1,v)dv}\Big) M_-(v) \quad , \quad \xi>0,$$

$$f_B(1,v) = \lambda'\Big(\frac{\int_{\xi>0}|\xi| f_B(1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv + \int_{\xi<0}|\xi| f_B(-1,v)dv}\Big) M_+(v) \quad , \quad \xi<0,$$

with

$$\lambda' = \frac{\lambda}{\int min(\mu, (1+|v|)^{\beta}) F_B(x,v)dxdv}.$$

Keeping, l, j, r, m, μ fixed, denote $f^{j,\alpha,l,r,m,\mu}$ by f^{α} and study the passage to the limit when α tends to 0. Writing the equations (2.21, 2.20) in exponential form and using the averaging lemmas together with a convolution with a mollifier ([2],[7]) give that f^{α}_A and F^{α} are strongly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$. Denote by f_A and F the limits of f^{α}_A and F^{α}. The passage to the limit when α tends to 0 in the equation (2.21) yields

$$\xi \frac{\partial}{\partial x} f_A = \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f_A}{1+\frac{F}{j}}(x,v') \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*^{'})dv_* d\omega$$

$$-f_A \int_{\mathbb{R}^3_{v_*}\times S} \chi^{r,m} B_{m,n,\mu} \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*)dv_* d\omega, \quad (x,v)\in(-1,1)\times\mathbb{R}^3_v,$$

$$f_A(-1,v) = \frac{\lambda}{\int min(\mu, (1+|v|)^{\beta}) F_A(x,v)dxdv} M_-(v) \quad , \quad \xi>0, \tag{2.22}$$

$$f_A(1,v) = \frac{\lambda}{\int min(\mu, (1+|v|)^{\beta}) F_A(x,v)dxdv} M_+(v) \quad , \quad \xi<0,$$

with

$$\int min(\mu, (1+|v|)^{\beta}) f_A(x,v)dxdv = 1.$$

For the same reasons, the limit f_B of f_B^α satisfies

$$\xi\frac{\partial}{\partial x}f_B=\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f_B}{1+\frac{F}{j}}(x,v')\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v^{'}_*)dv_*d\omega$$

$$-f_B\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3_v, \tag{2.23}$$

$$f_B(-1,v)=\sigma(-1)\lambda' M_-(v),\quad \xi>0,\quad f_B(1,v)=\sigma(1)\lambda' M_+(v),\quad \xi<0,$$

with

$$\int min(\mu,(1+|v|)^\beta)f_B(x,v)dxdv=1,$$

where

$$\sigma(-1)=\frac{\int_{\xi<0}|\xi|f_B(-1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv+\int_{\xi<0}|\xi|f_B(-1,v)dv},$$

$$\sigma(1)=\frac{\int_{\xi>0}\xi f_B(1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv+\int_{\xi<0}|\xi|f_B(-1,v)dv}$$

and

$$\lambda'=\frac{\lambda}{\int min(\mu,(1+|v|)^\beta)F_B(x,v)dxdv}.$$

The passage to the limit in (2.22) and in (2.23), when l tends to ∞ is similar and implies that the limits f_A and f_B of f_A^l and f_B^l are solutions to

$$\xi\frac{\partial}{\partial x}f_A=\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f_A}{1+\frac{F}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(x,v^{'}_*)dv_*d\omega$$

$$-f_A\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f}{1+\frac{f}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3_v, \tag{2.24}$$

$$f_A(-1,v)=k_AM_-(v),\quad \xi>0,\quad f_A(1,v)=k_AM_+(v),\quad \xi<0,$$

with

$$\int min(\mu,(1+v)^\beta)f_A(x,v)dxdv=1,$$

where k_A is defined in (2.21) before passing to the limit and

$$\xi\frac{\partial}{\partial x}f_B=\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f_B}{1+\frac{F}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(x,v^{'}_*)dv_*d\omega$$

$$-f_B\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\frac{f}{1+\frac{f}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3_v, \tag{2.25}$$

$$f_B(-1,v)=\sigma(-1)\lambda' M_-(v),\quad \xi>0,\quad f_B(1,v)=\sigma(1)\lambda' M_+(v),\quad \xi<0,$$

$$\int min(\mu,(1+|v|)^{\beta})f_B(x,v)dxdv=1,$$

with,

$$\begin{aligned}\sigma(-1) &= \frac{\int_{\xi<0}|\xi|f_B(-1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv+\int_{\xi<0}|\xi|f_B(-1,v)dv},\\ \sigma(1) &= \frac{\int_{\xi>0}\xi f_B(1,v)dv}{\int_{\xi>0}\xi f_B(1,v)dv+\int_{\xi<0}|\xi|f_B(-1,v)dv},\end{aligned}$$

and

$F=F_A+F_B$, with F_A and F_B defined by (2.1, 2.2) after passing to the limit.

We are going to pass to the limit when j tends to the infinity (2.24, 2.25). The sequences of solutions (f^j_A) and (f^j_B) to (2.24) and (2.25) are weakly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$. Indeed, $f^j=f^j_A+f^j_B$ satisfies the Boltzmann equation for a one component gas. By using the entropy production term ([2]), we find that f^j is weakly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$. The weak L^1 compactness of f^j_A and f^j_B follows from $0\leq f^j_A\leq f^j$ and $0\leq f^j_B\leq f^j$. Denote by f_A and f_B the limits of f^j_A and f^j_B, up to subsequences. Furthermore, $Q^+_j(f^j,f^j)$ and $Q^-_j(f^j,f^j)$ are weakly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$ ([2]). So, by the inequalities

$$\begin{gathered}Q^+_j(f^j_A,f^j)\leq Q^+_j(f^j,f^j), Q^+_j(f^j_B,f^j)\leq Q^+_j(f^j,f^j),\\ Q^-_j(f^j_A,f^j)\leq Q^-_j(f^j,f^j) \text{ and } Q^-_j(f^j_B,f^j)\leq Q^-_j(f^j,f^j),\end{gathered}$$

the collision terms $Q^+_j(f^j_B,f^j), Q^-_j(f^j_A,f^j)$, $Q^-_j(f^j_A,f^j)$ and $Q^-_j(f^j_B,f^j)$ are weakly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$. So, f^j_A and f^j_B satisfying (2.24, 2.25), $\xi\frac{\partial}{\partial x}f^j_A$ and $\xi\frac{\partial}{\partial x}f^j_B$ are weakly compact in $L^1([-1,1]\times\mathbb{R}^3_v)$. Pass to the limit when $j\rightarrow+\infty$ in the weak formulation of (2.25),

$$\begin{aligned}&\int_{[-1,1]\times\mathbb{R}^3_v}\xi\frac{\partial}{\partial x}\varphi(x,v)f^j_B(x,v)dxdv+\int_{[-1,1]\times\mathbb{R}^3_v}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\varphi(x,v)\\ &\quad(\frac{f^j_B}{1+\frac{F^j_A+F^j_B}{j}}(x,v')\frac{f^j}{1+\frac{f^j}{j}}(x,v'_*)-f^j_B(x,v)\frac{f^j}{1+\frac{f^j}{j}}(x,v_*))dv_*d\omega dxdv \qquad (2.26)\\ &\qquad=\int_{\xi>0}\xi f^j_B(1,v)\varphi(1,v)dv-\int_{\xi<0}\xi f^j_B(-1,v)\varphi(-1,v)dv\\ &\quad-\sigma^j(-1)\lambda'\int_{\xi>0}\xi M_-(v)\varphi(-1,v)dv+\sigma^j(1)\lambda'\int_{\xi<0}\xi M_+(v)\varphi(1,v)dv,\end{aligned}$$

where

$$\sigma^j(-1) = \frac{\int_{\xi<0}|\xi|f_B^j(-1,v)dv}{\int_{\xi<0}|\xi|f_B^j(-1,v)dv + \int_{\xi>0}\xi f_B^j(1,v)dv},$$
$$\sigma^j(1) = \frac{\int_{\xi>0}\xi f_B^j(1,v)dv}{\int_{\xi<0}|\xi|f_B^j(-1,v)dv + \int_{\xi>0}\xi f_B^j(1,v)dv},$$

and φ is a test function belonging to $C_c^1([-1,1]\times\mathbb{R}_v^3)$. First,

$$\lim_{j\to\infty}\int_{[-1,1]\times\mathbb{R}_v^3}\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}\varphi(x,v)\frac{f_B^j}{1+\frac{F_A^j+F_B^j}{j}}(x,v')\frac{f^j}{1+\frac{f^j}{j}}(x,v'_*)dxdvdv_*d\omega$$
$$= \lim_{j\to\infty}\int_{[-1,1]\times\mathbb{R}_v^3}\int_{\mathbb{R}_{v_*}^3\times S}\chi^{r,m}B_{m,n,\mu}\varphi(x,v')f_B^j(x,v)f^j(x,v_*)dxdvdv_*d\omega.$$

$\chi^{r,m}B_{m,n,\mu}$ being bounded, f^j and $\xi\frac{\partial}{\partial x}f^j$ being weakly compact in $L^1([-1,1]\times\mathbb{R}_{v_*}^3\times S)$, an averaging lemma applies, so that $(\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}f^j(x,v_*)\varphi(x,v')dv_*d\omega)_{j\in\mathbb{N}}$ converges to $\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}f(x,v_*)\varphi(x,v')dv_*d\omega$ in $L^1([-1,1]\times\mathbb{R}_v^3)$. Furthermore, because of the truncation in the ξ_* variable for small velocities, this sequence is bounded. f_B^j and $\xi\frac{\partial}{\partial x}f_B^j$ being weakly compact in $L^1([-1,1]\times\mathbb{R}_v^3)$, using once more an averaging lemma

$$(\int_{[-1,1]\times\mathbb{R}_v^3}\int_{\mathbb{R}_{v_*}^3\times S}\chi^{r,m}B_{m,n,\mu}f_B^j(x,v)f^j(x,v_*)\varphi(x,v')dv_*d\omega dxdv)_{j\in\mathbb{N}}$$

converges to

$$\int_{[-1,1]\times\mathbb{R}_v^3}\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}f_B(x,v)f(x,v_*)\varphi(x,v')dv_*d\omega dxdv.$$

So, $Q_j(f_B^j,f^j)$ converges to $Q(f_B,f)$ in L^1. Hence, reasoning as in the proof of Lemma 2.2, we find that $\sigma^j(-1)$ (resp $\sigma^j(1)$) tends to $\sigma(-1)$ (resp $\sigma(1)$), when j tends to infinity. By passing to the limit when j tends to infinity in (2.26), it holds that there is $f_B^{r,\mu}$ solution to

$$\xi\frac{\partial}{\partial x}f_B^{r,\mu} = \int_{\mathbb{R}_{v_*}^3\times S}\chi^{r,m}B_{m,n,\mu}f_B^{r,\mu}(x,v')f^{r,\mu}(x,v_*^{'})dv_*d\omega$$
$$-f_B^{r,\mu}\int_{\mathbb{R}_{v_*}^3\times\mathbb{S}^2}\chi^{r,m}B_{m,n,\mu}f^{r,\mu}(x,v_*)dv_*d\omega, \quad (x,v)\in(-1,1)\times\mathbb{R}_v^3, \qquad (2.27)$$
$$f_B^{r,\mu}(-1,v) = \sigma(-1)\lambda' M_-(v), \quad \xi>0, \quad f_B^{r,\mu}(1,v)=\sigma(1)\lambda' M_+(v), \quad \xi<0,$$

with

$$\int min(\mu,(1+|v|)^{\beta}) f_B^{r,\mu}(x,v)dxdv = 1.$$

Here,

$$\sigma(-1) = \frac{\int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv}{\int_{\xi>0} \xi f_B^{r,\mu}(1,v)dv + \int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv}$$

and

$$\sigma(1) = \frac{\int_{\xi>0} \xi f_B^{r,\mu}(1,v)dv}{\int_{\xi>0} \xi f_B^{r,\mu}(1,v)dv + \int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv}.$$

Moreover,

$$\int_{\mathbb{R}^3_v} \xi f_B^{r,\mu}(1,v)dv - \int_{\mathbb{R}^3_v} \xi f_B^{r,\mu}(-1,v)dv = 0,$$

so that,

$$\int_{\xi>0} |\xi| f_B^{r,\mu}(1,v)dv + \int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv = \lambda'.$$

Hence, the boundary conditions in (2.27) write

$$\begin{aligned} f_B^{r,\mu}(-1,v) &= \int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv M_-(v) \quad , \quad \xi>0, \\ f_B^{r,\mu}(1,v) &= \int_{\xi>0} |\xi| f_B^{r,\mu}(1,v)dv M_+(v) \quad , \quad \xi<0. \end{aligned}$$

The passage to the limit when $m \to +\infty$ and $n \to +\infty$ in the equation (2.27) is performed as before and implies that there is $f_B^{r,\mu}$ satisfying,

$$\xi \frac{\partial}{\partial x} f_B^{r,\mu} = \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^r B_\mu(v-v_*,\omega) f_B^{r,\mu}(x,v') f^{r,\mu}(x,v_*^{'}) dv_* d\omega$$

$$-f_B^{r,\mu} \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^r B_\mu(v-v_*,\omega) f^{r,\mu}(x,v_*) dv_* d\omega, \quad (x,v)\in(-1,1)\times\mathbb{R}^3_v,$$

$$f_B^{r,\mu}(-1,v) = M_-(v) \int_{\xi<0} |\xi| f_B^{r,\mu}(-1,v)dv \quad , \quad \xi>0, \tag{2.28}$$

$$f_B^{r,\mu}(1,v) = M_+(v) \int_{\xi>0} \xi f_B^{r,\mu}(1,v)dv \quad , \quad \xi<0,$$

with

$$\int min(\mu,(1+|v|)^{\beta})f_B^{r,\mu}(x,v)dxdv=1.$$

For the same reasons, there is $f_A^{r,\mu}$ satisfying,

$$\xi\frac{\partial}{\partial x}f_A^{r,\mu}=\int_{\mathbb{R}^3_v\times\mathbb{S}^2}\chi^r B_\mu(v-v_*,\omega)f_A^{r,\mu}(x,v')f^{r,\mu}(x,v_*^{'})dv_*d\omega$$

$$-f_A^{r,\mu}\int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2}\chi^r B_\mu(v-v_*,\omega)f^{r,\mu}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3_v,$$

$$f_A^{r,\mu}(-1,v)=k_AM_-(v),\xi>0,\quad f_A^{r,\mu}(1,v)=k_AM_+(v),\xi<0, \tag{2.29}$$

with

$$\int min(\mu,(1+|v|)^{\beta})f_A^{r,\mu}(x,v)dxdv=1,$$

where k_A is defined in the equation (2.21) before passing to the limit.

3 End of the proof of the main theorem

Let $(r_j)_{j\in\mathbb{N}}$ with $\lim_{j\to+\infty}r_j=0$ and $(\mu_j)_{j\in\mathbb{N}}$ with $\lim_{j\to+\infty}\mu_j=+\infty$, $f_A^j=f_A^{r_j,\mu_j}$ and $f_B^j=f_B^{r_j,\mu_j}$. The passage to the limit when $j\to+\infty$ is now performed in the weak formulations satisfied by f_A^j and f_B^j. A positive number δ being fixed, let φ be a test function vanishing for $|\delta|\leq\delta$ and for $|v|\geq\frac{1}{\delta}$. Since $f^j=f_A^j+f_B^j$ satisfies the Boltzmann equation for a one component gas, using the entropy production term ([1]), f^j, $Q_j^-(f^j,f^j)$ and $Q_j^+(f^j,f^j)$ are weakly compact in $L^1([-1,1]\times\{v\in\mathbb{R}^3;|\xi|\geq\delta,|v|\leq\frac{1}{\delta}\})$. The weak compactness of f_A^j, f_B^j, $Q_j^-(f_A^j,f^j)$, $Q_j^+(f_A^j,f^j)$, $Q_j^-(f_B^j,f^j)$ and $Q_j^+(f_B^j,f^j)$ follows from the inequalities

$$0\leq f_A^j\leq f^j,\quad 0\leq f_B^j\leq f^j,\quad Q_j^-(f_A^j,f^j)\leq Q_j^-(f^j,f^j),$$
$$Q_j^+(f_A^j,f^j)\leq Q_j^+(f^j,f^j),\quad Q_j^-(f_B^j,f^j)\leq Q_j^-(f^j,f^j)$$
$$\text{and } Q_j^+(f_B^j,f^j)\leq Q_j^+(f^j,f^j).$$

Denote by f_A and f_B the limits of f_A^j and f_B^j, up to subsequences. As in ([1],[2]),

$$\lim_{j\to\infty}\int[Q_j^-(f_A^j,f^j)-Q^-(f_A,f)]\varphi dxdv=0. \tag{3.1}$$

Next, performing the change of variable $(v, v_*, \omega) \mapsto (v', v'_*, -\omega)$, the same property is obtained for the gain term,

$$\lim_{j\to+\infty} \int_{-1}^{1} \int_{\mathbb{R}^3_v} Q^+_j(f^j_A, f^j)\varphi dxdv = \int_{-1}^{1} \int_{\mathbb{R}^3_v} Q^+(f_A, f)\varphi dxdv.$$

Analogously, the same result holds for f_B. It remains to pass to the limit in the boundary terms (1.4) i.e to prove the weak convergence in $L^1(\{v \in \mathbb{R}^3_v, \xi > 0\})$ (resp $L^1(\{v \in \mathbb{R}^3_v, \xi < 0\})$) of $f^j_B(1,.)$ (resp. $f^j_B(-1,.)$) to $f_B(1,.)$ (resp. $f_B(-1,.)$). First, the fluxes $\int_{\xi>0} \xi f^j_B(1,v)dv$ and $\int_{\xi<0} |\xi| f^j_B(-1,v)dv$ are controled in the following way. From (2.28) written in the exponential form, it holds that

$$\begin{aligned}
f^j_B(x,v) &\geq f^j_B(-1,v)e^{-\int^0_{-\frac{1+x}{\xi}} \int_{\mathbb{R}^3_{v_*}\times S} \chi^j B^j f^j(x+s\xi,v_*)dv_*d\omega ds}, \quad \xi > \frac{1}{2}, |v| \leq 2,\\
f^j_B(x,v) &\geq f^j_B(1,v)e^{-\int^0_{\frac{1-x}{\xi}} \int_{\mathbb{R}^3_{v_*}\times S} \chi^j B^j f^j(x+s\xi,v_*)dv_*d\omega ds}, \quad \xi < -\frac{1}{2}, |v| \leq 2.
\end{aligned} \qquad (3.2)$$

Recall that,

$$\nu^j(x,v) = \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2} \chi^j B^j f^j(x,v_*)dv_*d\omega.$$

For v satisfying $|v| \leq 2$ with $\xi > \frac{1}{2}$ or $\xi < -\frac{1}{2}$, $\int_{-1}^{1} \frac{\nu^j(z,v)}{|\xi|}dz$ is uniformly bounded from above. Hence, using the definition of the boundary conditions (1.4) in (3.2),

$$\begin{aligned}
f^j_B(x,v) &\geq cM_-(v) \int_{\xi<0} |\xi| f^j_B(-1,v)dv \quad , \quad \xi > \frac{1}{2}, \quad |v| \leq 2,\\
f^j_B(x,v) &\geq cM_+(v) \int_{\xi>0} \xi f^j_B(1,v)dv \quad , \quad \xi < -\frac{1}{2}, \quad |v| \leq 2.
\end{aligned}$$

So,

$$\begin{aligned}
&c \int_{\{\xi>\frac{1}{2},|v|\leq 2\}\cup\{\xi<-\frac{1}{2},|v|\leq 2\}} f^j_B(x,v)dxdv\\
&\geq \int_{\xi>0} \xi f^j_B(1,v)dv + \int_{\xi<0} |\xi| f^j_B(-1,v)dv.
\end{aligned}$$

f^j_B being non negative,

$$\begin{aligned}
&c \int_{-1}^{1} \int_{\mathbb{R}^3_v} min(\mu, (1+|v|)^\beta) f^j_B(x,v)dxdv\\
&\geq \int_{\xi>0} \xi f^j_B(1,v)dv + \int_{\xi<0} |\xi| f^j_B(-1,v)dv.
\end{aligned}$$

Since $\int_{-1}^{1}\int_{\mathbb{R}^3_v} min(\mu,(1+|v|)^\beta) f_B^j(x,v)dxdv = 1$, the fluxes $\int_{\xi>0}\xi f_B^j(1,v)dv$ and $\int_{\xi<0}|\xi| f_B^j(-1,v)dv$ are bounded uniformly w.r.t j.
Furthermore, the energy fluxes are controlled. Indeed, by conservation of the energy for f^j,

$$\begin{aligned}&\int_{\xi>0}\xi v^2 f_B^j(1,v)dv + \int_{\xi<0}|\xi| v^2 f_B^j(-1,v)dv\\ \leq &\int_{\xi>0}\xi v^2 f^j(-1,v)dv + \int_{\xi<0}|\xi| v^2 f^j(1,v)dv.\end{aligned}$$

By definition of the boundary conditions (2.29) and (2.28),

$$\begin{aligned}&\int_{\xi>0}\xi v^2 f_B^j(1,v)dv + \int_{\xi<0}|\xi| v^2 f_B^j(-1,v)dv\\ \leq &(k^j + \int_{\xi'<0}|\xi'| f_B^j(-1,v')dv')\int_{\xi>0}\xi v^2 M_-(v)dv\\ &+(k^j + \int_{\xi'>0}\xi' f_B^j(1,v')dv')\int_{\xi<0}|\xi| v^2 M_+(v)dv.\end{aligned} \tag{3.3}$$

The right-hand side of (3.3) being bounded,

$$\int_{\xi>0}\xi v^2 f_B^j(1,v)dv + \int_{\xi<0}|\xi| v^2 f_B^j(-1,v)dv \leq c.$$

Finally, the entropy fluxes are controled. Indeed, $f^j = f_A^j + f_B^j$ satisfies the following equation

$$\xi\frac{\partial}{\partial x}\big(f^j(log(f^j)-1)\big) = Q_j(f^j,f^j)log(f^j). \tag{3.4}$$

Using a Green's formula and an entropy estimate in (3.4), leads to

$$\begin{aligned}&\int_{\xi>0}\xi f_B^j(1,v)\log f_B^j(1,v)dv + \int_{\xi<0}|\xi| f_B^j(-1,v)\log f_B^j(-1,v)dv\\ &\qquad\leq (\int_{\xi'>0}\xi' f_B^j(1,v')dv' + k^j)\\ &\int_{\xi<0}|\xi| M_+(v) log(M_+(v)(\int_{\xi'>0}\xi' f_B^j(1,v')dv' + k^j))dv\\ &\qquad+(\int_{\xi'<0}|\xi'| f_B^j(-1,v')dv' + k^j)\\ &\int_{\xi>0} M_-(v)\log(M_-(v)(\int_{\xi'<0}|\xi'| f_B^j(-1,v')dv' + k^j))dv.\end{aligned}$$

By the Dunford-Pettis criterion ([5]), $f_B^j(1,.)$ is weakly compact in $L^1(\{v\in\mathbb{R}^3_v,\xi>0\})$. Let one of its subsequence still denoted by $f_B^j(1,.)$, converging weakly to some g_+ in $L^1(\{v\in\mathbb{R}^3_v,\xi>0\})$. It remains to prove that $g_+=f_B(1,.)$. Consider a test function φ vanishing on $\{|\xi|\leq\delta\}\cup\{|v|\geq\frac{1}{\delta}\}$ and satisfying $\varphi(x,v)=\varphi_1(x)\varphi_2(v)$ with $\varphi_1=1$ in a neighborhood of 1. Recall that the trace $f_B(1,v)$ can be defined by

$$f_B(1,v)=\lim_{\epsilon_0\rightarrow 0}\frac{1}{\epsilon_0}\int_0^{\epsilon_0}f_B(1-\epsilon,v)d\epsilon\quad([4]).$$

(φf_B^j) satisfies the equation

$$\xi\frac{\partial(\varphi f_B^j)}{\partial x}=\xi\frac{\partial\varphi}{\partial x}f_B^j+Q_j(f_B^j,f^j)\varphi. \tag{3.5}$$

Integrating (3.5) on $[1-\varepsilon,1]\times\mathbb{R}^3_v$ and using a Green's formula, it holds that

$$\begin{aligned}&|\frac{1}{\epsilon_0}\int_{\mathbb{R}^3_v}\int_0^{\epsilon_0}(f_B^j(1,v)-f_B^j(1-\epsilon,v))\varphi_2(v)dvd\epsilon|\\ \leq&\frac{1}{\epsilon_0}\int_0^{\epsilon_0}\int_{\mathbb{R}^3_v}\int_{1-\epsilon_0}^1|Q_j(f_B^j,f^j)(x,v)\varphi(x,v)|dxdvd\epsilon\\ &+\frac{1}{\epsilon_0}\int_0^{\epsilon_0}\int_{\mathbb{R}^3_v}\int_{1-\epsilon_0}^1|f_B^j(x,v)\xi\frac{\partial}{\partial x}\varphi(x,v)|dxdvd\epsilon.\end{aligned} \tag{3.6}$$

Let $\eta>0$ be given. By the weak compactness of f_B^j and $Q_j(f_B^j,f^j)$ in $L^1([-1,1]\times\{v\in\mathbb{R}^3_v,|\xi|\geq\delta,|v|\leq\frac{1}{\delta}\})$, there exists $\tilde{\epsilon_0}>0$ such that for $\epsilon_0<\tilde{\epsilon_0}$, uniformly w.r.t j,

$$\int_{\mathbb{R}^3_v}\int_{1-\epsilon_0}^1|Q_j(f_B^j,f^j)(x,v)\varphi(x,v)|dxdv<\frac{\eta}{2},$$
$$\int_{\mathbb{R}^3_v}\int_{1-\epsilon_0}^1|f_B^j(x,v)\xi\frac{\partial}{\partial x}\varphi(x,v)|dxdv<\frac{\eta}{2}.$$

So the inequality (3.6) gives, for $\epsilon_0<\tilde{\epsilon_0}$, uniformly w.r.t j,

$$|\frac{1}{\epsilon_0}\int_{\mathbb{R}^3_v}\int_0^{\epsilon_0}\xi(f_B^j(1,v)-f_B^j(1-\epsilon,v))\varphi_2(v)d\epsilon dv|\leq\eta. \tag{3.7}$$

By the weak compactness of f_B^j in $L^1([-1,1]\times\{v\in\mathbb{R}^3_v,|\xi|\geq\delta,|v|\leq\frac{1}{\delta}\})$,

$$\int_{\mathbb{R}^3_v}\int_0^{\epsilon_0} f_B^j(1-\epsilon,v)\varphi_2(v)dvd\epsilon \longrightarrow \int_{\mathbb{R}^3_v}\int_0^{\epsilon_0} f_B(1-\epsilon,v)\varphi_2(v)dvd\epsilon$$

when j tends to infinity. Passing to the limit when j tends to infinity in the inequality (3.7), implies that

$$\frac{1}{\epsilon_0}|\int_{\mathbb{R}^3_v}\int_0^{\epsilon_0}\xi(g_+(1,v)-f_B(1-\epsilon,v))\varphi_2(v)dvd\epsilon|\leq\eta.$$

Next, passing to the limit when ϵ_0 tends to 0,

$$|\int_{\mathbb{R}^3_v}\xi(g_+(1,v)-f_B(1,v))\varphi_2(v)dv|\leq\eta. \tag{3.8}$$

The inequality (3.8) being true for all $\eta>0$, g_+ and $f_B(1,.)$ are equal on the sets $\{v\in\mathbb{R}^3_v, |\xi|\geq\delta, |v|\leq\frac{1}{\delta}\}$ for all $\delta>0$ and so *a.e.* The same result holds for $f_B(-1,.)$. Therefore, we can pass to the limit in (2.28, 2.29). This ends the proof of Theorem 1.1. □

4 Some extensions

Theorem 1.1 can be generalized to the case of any values of the β-norms, M_A and M_B.

Corollary 4.1. *Given β with $0\leq\beta<2$, $M_A>0$ and $M_B>0$, there is a weak solution to the stationary problem with β-norms M_A and M_B.*

Proof of Corollary 4.1. Define $M=M_A+M_B$. In the first part of the proof of Theorem 1.1, choose λ satisfying,

$$\lambda\geq min(\frac{M_A}{\int_{\xi>0}M_-(v)min(\mu,(1+|v|)^\beta)e^{-\frac{2+M}{\xi}}dv};\frac{M_A}{\int_{\xi<0}M_+(v)min(\mu,(1+|v|)^\beta)e^{-\frac{2+M}{|\xi|}}dv})$$

and

$$\lambda\geq min(\frac{M_B}{\int_{\xi>0}M_-(v)min(\mu,(1+|v|)^\beta)e^{-\frac{2+M}{\xi}}dv};\frac{M_B}{\int_{\xi<0}M_+(v)min(\mu,(1+|v|)^\beta)e^{-\frac{2+M}{|\xi|}}dv}).$$

In the fixed point part, choose

$$K = \{f \in L^1_+([-1,1] \times \mathbb{R}^3); \int min(\mu, (1+|v|)^\beta) f(x,v) dx dv = M\}. \quad (4.1)$$

Reasoning as before, there are f_A and f_B solutions to the equation (1.1) with respective $\beta-$norms M_A and M_B. $\square$

In the presence of several gases of the A component and several gases of the B component, Corollary 4.1 can be generalized. Consider $(f_{A_1}....f_{A_{N_A}}, f_{B_1}....f_{B_{N_B}})$ with f_{A_i} satisfying

$$\frac{\partial}{\partial x} f_{A_i} = Q(f_{A_i}, f_{A_1}) + ... + Q(f_{A_i}, f_{A_{N_A}}) + Q(f_{A_i}, f_{B_1}) + ... + Q(f_{A_i}, f_{B_{N_B}}),$$
$$(x,v) \in (-1,1) \times \mathbb{R}^3_v,$$
$$f_{A_i}(-1,v) = k_{A_i} M_-(v), \quad \xi > 0, \quad f_{A_i}(1,v) = k_{A_i} M_+(v), \quad \xi < 0,$$

and f_{B_i} satisfying

$$\frac{\partial}{\partial x} f_{B_i} = Q(f_{B_i}, f_{A_1}) + ... + Q(f_{B_i}, f_{A_{N_A}}) + Q(f_{B_i}, f_{B_1}) + ... + Q(f_{B_i}, f_{B_{N_B}}),$$
$$(x,v) \in (-1,1) \times \mathbb{R}^3_v,$$
$$f_{B_i}(-1,v) = (\int_{\xi'<0} |\xi'| f_{B_i}(-1,v') dv') M_-(v) \quad , \quad \xi > 0,$$
$$f_{B_i}(1,v) = (\int_{\xi'>0} \xi' f_{B_i}(1,v') dv') M_+(v) \quad , \quad \xi < 0.$$

Corollary 4.2. *Given β with $0 \leq \beta < 2$, $M_{A_1}....M_{A_{N_A}}$ and $M_{B_1}....M_{B_{N_B}}$ there are weak solutions $f_{A_1}...f_{B_{N_B}}$ to the stationary problem with respective β-norms $M_{A_1}, ..., M_{B_{N_B}}$.*

Proof of Corollary 4.2. Let $f = f_{A_1} + ... + f_{A_N} + f_{B_1} + ... + f_{B_N}$. The Boltzmann operator corresponding to the A_i component is
$Q(f_{A_i}, f) = Q(f_{A_i}, f_{A_1}) + ... + Q(f_{A_i}, f_{A_{N_A}}) + Q(f_{A_i}, f_{B_1}) + ... + Q(f_{A_i}, f_{B_{N_B}})$, ([6]). For the fixed point step, define K as in (4.1) and consider the compact and continuous map T from the closed convex set $K \times [0,1]^{N_B}$ into itself,

$$T : (f, \theta_1, ..., \theta_{N_B}) \mapsto (f_{A_1} + ... + f_{A_{N_A}} + f_{B_1} ... f_{B_{N_B}}, \tilde{\theta}_1, ..., \tilde{\theta}_{N_B}),$$

with

$$\tilde{\theta}_{N_i} = \frac{\int_{\xi<0} |\xi| f_{B_{N_i}}(-1,v) dv}{\int_{\xi<0} |\xi| f_{B_{N_i}}(-1,v) dv + \int_{\xi>0} \xi f_{B_{N_i}}(1,v) dv}.$$

The corollary follows by the same arguments as before. □

Remark that the case of a one component gas with one boundary condition of the type (1.3) and another of the type (1.4) can also be solved. It comes back to the diffuse-reflection problem solved in ([2]). Furthermore, this problem can be generalized to several components by reasoning as in the proof of Corollary 4.2.
Theorem 1.1 can also be generalized to the case of a convex combination of boundary conditions of the type (1.3) and (1.4),

$$\xi \frac{\partial}{\partial x} f = Q(f,f), \quad (x,v) \in (-1,1) \times \mathbb{R}^3_v,$$
$$f(-1,v) = a\Big(\int_{\xi<0} |\xi| f(-1,v)dv\Big) M_-(v) + (1-a)kM_-(v) \quad , \quad \xi > 0,$$
$$f(1,v) = a\Big(\int_{\xi>0} \xi f(1,v)dv\Big) M_+(v) + (1-a)kM_+(v) \quad , \quad \xi < 0, \tag{4.2}$$
$$a \in [0,1].$$

Corollary 4.3. *Given β with $0 \leq \beta < 2$, $M > 0$ there is a weak solution to the stationary problem (4.2) with the β-norm M.*

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