

CONSTRUCTION AND APPROXIMATION OF THE POLYATOMIC BITEMPERATURE EULER SYSTEM

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ABSTRACT. This paper is devoted to the study of the bitemperature Euler system in a polyatomic setting. Physically, this model describes a mixture of one species of ions and one species of electrons in the quasi-neutral regime. We firstly derive the model starting from a kinetic polyatomic model and performing next a fluid limit. This kinetic model is shown to satisfy fundamental properties. Finally, a numerical scheme is derived and tested.

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1. Introduction. This work is devoted to the modelling and the numerical approximation of the non conservative bitemperature Euler system in the context of plasma physics. Physically, this model describes the interaction of one species of ions and one species of electrons, under the quasi-neutrality assumption.

The aim of this paper is more precisely to provide a construction and an approximation of the polyatomic Euler bitemperature system. In ([2]), the model was derived from a kinetic monoatomic system by using an hydrodynamic limit. Moreover, this construction leads to a kinetic numerical scheme. However relaxation schemes and discrete BGK schemes have been developped in the polytropic situation for a general γ law. In the present paper, this derivation is based again on an hydrodynamic limit performed from an underlying polyatomic kinetic model. In the present case, the unknowns of the kinetic equations are the distribution functions $f(t, x, v, I)$ depending on time t , space x , velocity v and of one-dimensionnal internal energy parameter $I > 0$. This energy parameter collects translational, vibrational and rotational energies. Kinetic models with continuous energy variables have been introduced in ([4]) where the motivation was to develop a Monte-Carlo method. In ([5]), the authors derived a mesoscopic model of Boltzmann type associated to the previous microscopic model. This collision operator satisfies fundamental properties (H theorem, ...). The generalisation to mixtures has been performed in ([9]). For some applications of such models we refer to ([16], [18], [12]). In ([16]) the authors analyse the shock wave structure of some polyatomic gases. So by using an ESBGK model ([1], [7]) they show the presence of a double layer structure that is specific to the polyatomic setting. We mention that in ([10], [13]), the authors develop polyatomic models with a discrete internal energy variable.

As observed in ([2], [17]), it is possible to find an underlying kinetic model where the force term is related to the nonconservative terms. One advantage of the kinetic model, is its conservative form. In the present paper, the kinetic model describing the ion and electron interaction is a two component polyatomic BGK model based on one continous internal energy variable ([9]) coupled with Ampère and Poisson equations. Hence starting from a standard semi-discretization of this model, the Chapman-Engskog procedure leads to a numerical scheme for the bitemperature model.

The plan of the paper is the following. Section 2 deals with the presentation of the different models that are used in this paper. In particular, we prove the hyperbolicity of the bitemperature system. In section 3, the fluid model is obtained starting from the polyatomic model. In section 4 exact solutions for the model are computed and in section 5, the numerical scheme is developped. The last part is devoted to the numerical results where the scheme is tested and validated.

2. The mathematical models.

2.1. The bitemperature Euler system.

2.1.1. *The fluid model.* The bitemperature Euler system writes

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) = 0, \\ \partial_t \mathcal{E}_e + \partial_x(u(\mathcal{E}_e + p_e)) - u(c_i \partial_x p_e - c_e \partial_x p_i) = \nu_{ei}(T_i - T_e), \\ \partial_t \mathcal{E}_i + \partial_x(u(\mathcal{E}_i + p_i)) + u(c_i \partial_x p_e - c_e \partial_x p_i) = -\nu_{ei}(T_i - T_e), \end{cases} \quad (1)$$

where $\rho \geq 0$ represents the total density of the plasma, u is the average velocity of the plasma, \mathcal{E}_e and \mathcal{E}_i are the total electronic and ionic energies. T_e and T_i represent the electronic and ionic temperatures.

One has $\rho = \rho_e + \rho_i$ where $\rho_e = n_e m_e$, $\rho_i = n_i m_i$ are electronic and ionic densities, electronic and ionic concentrations n_e and n_i being assumed to be linked by $Z = n_e/n_i \geq 1$. Z is considered as constant. This situation corresponds to the quasi-neutral regime. m_e and m_i represent the electronic and ionic masses. The mass fractions

$$c_\beta = \frac{\rho_\beta}{\rho}, \quad \beta = e, i \quad (2)$$

are then constant and c_e and c_i write

$$c_e = \frac{Z m_e}{m_i + Z m_e}, \quad c_i = 1 - c_e. \quad (3)$$

The total energies are linked to the internal electronic and ionic energies by

$$\mathcal{E}_\beta = \rho_\beta \varepsilon_\beta + \frac{1}{2} \rho_\beta u^2, \quad \beta \in \{e, i\}.$$

The electronic and ionic pressures and temperatures are related by $p_e = n_e k_B T_e$ and $p_i = n_i k_B T_i$. The electronic and ionic internal energies are then given by

$$\varepsilon_e = \frac{1}{(\gamma_e - 1)m_e} k_B T_e, \quad \varepsilon_i = \frac{1}{(\gamma_i - 1)m_i} k_B T_i,$$

where $\gamma_e, \gamma_i \in [1, 3]$, and k_B is the Boltzmann constant.

In the following we denote $\mathcal{U} = (\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i)$, $U_\beta = (\rho_\beta, \rho_\beta u, \mathcal{E}_\beta)$.

2.1.2. *Hyperbolicity of the system.* The system is rewritten by using the variables $\mathcal{V} = (\rho, u, \varepsilon_e, \varepsilon_i)$:

$$\begin{cases} \partial_t \rho + u \partial_x \rho + \rho \partial_x u = 0, \\ \partial_t u + u \partial_x u + \rho^{-1} \partial_\rho (p_e + p_i) \partial_x \rho + \rho^{-1} \partial_{\varepsilon_e} p_e \partial_x \varepsilon_e + \rho^{-1} \partial_{\varepsilon_i} p_i \partial_x \varepsilon_i = 0, \\ \partial_t \varepsilon_e + u \partial_x \varepsilon_e + \rho_e^{-1} p_e \partial_x u = \rho_e^{-1} \nu_{ei} (T_i - T_e), \\ \partial_t \varepsilon_i + u \partial_x \varepsilon_i + \rho_i^{-1} p_i \partial_x u = \rho_i^{-1} \nu_{ei} (T_e - T_i). \end{cases} \quad (4)$$

The matrix of the system (4) writes

$$A(\mathcal{V}) = u I_4 + \begin{pmatrix} 0 & \rho & 0 & 0 \\ \rho^{-1} \partial_\rho (p_e + p_i) & 0 & \rho^{-1} \partial_{\varepsilon_e} p_e & \rho^{-1} \partial_{\varepsilon_i} p_i \\ 0 & \rho_e^{-1} p_e & 0 & 0 \\ 0 & \rho_i^{-1} p_i & 0 & 0 \end{pmatrix}. \quad (5)$$

We get 4 eigenvalues

$$\lambda_1 = u - a, \quad \lambda_2 = \lambda_3 = u, \quad \lambda_4 = u + a,$$

where

$$a = \sqrt{\sum_{\beta=e,i} \gamma_\beta (\gamma_\beta - 1) c_\beta \varepsilon_\beta}. \quad (6)$$

The value of a given by (6) corresponds to the global sound velocity which yields for the classical Euler system $\sqrt{\gamma p/\rho}$. The eigenvectors associated to the eigenvalue u are equal to

$$r_2 = \begin{pmatrix} 0 \\ 0 \\ -(\gamma_i - 1)c_i \\ (\gamma_e - 1)c_e \end{pmatrix}, \quad r_3 = \begin{pmatrix} -\rho \\ 0 \\ \varepsilon_e \\ \varepsilon_i \end{pmatrix}.$$

This system is then hyperbolic diagonalisable. Let us show that the fields 2 and 3 linearly degenerated. A first computation gives $\lambda'_2(\mathcal{V}) = (0, 1, 0, 0)$. So

$$\lambda'_k(\mathcal{V}) \cdot r_k(\mathcal{V}) = 0, \quad k = 2, 3.$$

Consider now the fields 1 and 4. The eigenvectors are

$$r_1(\mathcal{V}) = \begin{pmatrix} -\rho \\ a \\ -(\gamma_e - 1)\varepsilon_e \\ -(\gamma_i - 1)\varepsilon_i \end{pmatrix}, \quad r_4(\mathcal{V}) = \begin{pmatrix} \rho \\ a \\ (\gamma_e - 1)\varepsilon_e \\ (\gamma_i - 1)\varepsilon_i \end{pmatrix}$$

and

$$\lambda'_1(\mathcal{V}) \cdot r_1(\mathcal{V}) = \lambda'_4(\mathcal{V}) \cdot r_4(\mathcal{V}) = \frac{1}{2a} \sum_{\beta=e,i} (\gamma_\beta(\gamma_\beta - 1)(\gamma_\beta + 1)c_\beta \varepsilon_\beta) > 0. \quad (7)$$

Hence the fields 1 and 4 are genuinely nonlinear.

2.2. The kinetic model.

2.2.1. *Notations.* Kinetic models are described by the distribution function f_β of species β depending on time $t \in \mathbb{R}^+$, space $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$ and on internal energy $I \in \mathbb{R}_+$.

Hydrodynamic quantities of species β are defined for $\alpha_\beta \geq 0$ by

$$n_\beta = \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_\beta I^{\alpha_\beta} dv dI, \quad u_\beta = \frac{1}{n_\beta} \int_{\mathbb{R}^3 \times \mathbb{R}_+} v f_\beta I^{\alpha_\beta} dv dI, \quad \varepsilon_\beta = \int_{\mathbb{R}^3 \times \mathbb{R}_+} (m_\beta \frac{v^2}{2} + I) f_\beta I^{\alpha_\beta} dv dI.$$

Velocities and temperatures of the mixture u et T are defined by

$$u = \frac{1}{\rho} (\rho_e u_e + \rho_i u_i), \quad (8)$$

$$T = \frac{((\frac{5}{2} + \alpha_e)n_e k_B + (\frac{5}{2} + \alpha_i)n_i k_B + \frac{1}{2}\rho_e u_e^2 + \frac{1}{2}\rho_i u_i^2 - \frac{1}{2}\rho u^2)}{(\frac{5}{2} + \alpha_e)n_e k_B + (\frac{5}{2} + \alpha_i)n_i k_B}. \quad (9)$$

The parameters α_e and α_i are related to γ_e and γ_i by the formula

$$\gamma_e = \frac{1}{\frac{5}{2} + \alpha_e} + 1, \quad \gamma_i = \frac{1}{\frac{5}{2} + \alpha_i} + 1.$$

For example in the diatomic case, we have $\alpha_e = \alpha_i = 0$.

Define the entropy of the mixture by

$$\mathcal{H}(f_e, f_i) = \mathcal{H}_s(f_e) + \mathcal{H}_s(f_i), \quad \text{with} \quad \mathcal{H}_s(f_\beta) = \int_{\mathbb{R}^3} (f_\beta \ln(f_\beta) - f_\beta) I^{\alpha_\beta} dv dI \quad (10)$$

and the entropy flux by

$$\Phi(f_e, f_i) = \Phi_s(f_e) + \Phi_s(f_i), \quad \text{with} \quad \Phi_s(f_\beta) = \int_{\mathbb{R}^3} v (f_\beta \ln(f_\beta) - f_\beta) I^{\alpha_\beta} dv dI. \quad (11)$$

2.2.2. *A polyatomic BGK model.* In this section we consider the following kinetic model for $\beta \in \{e; i\}$,

$$\begin{aligned} \partial_t f_\beta(t, x, v, I) + v \cdot \nabla f_\beta(t, x, v, I) + \frac{q_\beta}{m_\beta} E \cdot \nabla f_\beta(t, x, v, I) &= \frac{1}{\tau_\beta} (\mathcal{M}_\beta - f_\beta(t, x, v, I)) \\ &+ \frac{1}{\tau_{\beta\delta}} (\overline{\mathcal{M}_\beta}(f_\beta, f_\delta) - f_\beta(t, x, v, I)), \end{aligned} \quad (12)$$

with

$$\mathcal{M}_\beta = \frac{n_\beta}{(2\pi \frac{k_B}{m_\beta} T_\beta)^{\frac{3}{2}}} \frac{1}{\mathcal{Q}_\beta(T_\beta)} \exp\left(-\frac{(v - u_\beta)^2}{2k_B \frac{T_\beta}{m_\beta}} - \frac{I}{k_B T_\beta}\right), \quad (13)$$

$$\overline{\mathcal{M}_\beta}(f_e, f_i) = \frac{n_\beta}{(2\pi \frac{k_B}{m_\beta} T^\#)^{\frac{3}{2}}} \frac{1}{\mathcal{Q}_\beta(T^\#)} \exp\left(-\frac{(v - u^\#)^2}{2k_B \frac{T^\#}{m_\beta}} - \frac{I}{k_B T^\#}\right), \quad (14)$$

where

$$\mathcal{Q}_\beta(T) = \int_0^{+\infty} I^{\alpha_\beta} \exp(-\frac{I}{T}) dI, \quad \alpha_\beta \geq 0$$

and

$$u^\# = \frac{\frac{1}{\tau_{ei}} \rho_e u_e + \frac{1}{\tau_{ie}} \rho_i u_i}{\frac{1}{\tau_{ei}} \rho_e + \frac{1}{\tau_{ie}} \rho_i}, \quad (15)$$

$$T^\# = \frac{\frac{1}{\tau_{ei}} (\frac{5}{2} + \alpha_e) n_e T_e + \frac{1}{\tau_{ie}} (\frac{5}{2} + \alpha_i) n_i T_i + \frac{1}{2} (\frac{1}{\tau_{ei}} \rho_e u_e^2 + \frac{1}{\tau_{ie}} \rho_i u_i^2 - (\frac{1}{\tau_{ei}} \rho_e + \frac{1}{\tau_{ie}} \rho_i) (u^\#)^2)}{\frac{1}{\tau_{ei}} (\frac{5}{2} + \alpha_e) n_e + \frac{1}{\tau_{ie}} (\frac{5}{2} + \alpha_i) n_i}. \quad (16)$$

As suggested in ([14]) and developed in ([2]) the definitions (15, 16) of $u^\#$ and $T^\#$ allow to consider a model such that the mass ratios of the species are different. This situation corresponds to the case $\tau_{ei} \neq \tau_{ie}$.

The model (12, 13, 14, 15, 16) is coupled to Ampère and Poisson equations through the electric field E as

$$\partial_t E = -\frac{j}{\varepsilon_0}, \quad (17)$$

$$\nabla_x \cdot E = \frac{\bar{\rho}}{\varepsilon_0}. \quad (18)$$

j represents the plasmas current, $\bar{\rho}$ the total charge and ε_0 is the permittivity. j and $\bar{\rho}$ are defined by

$$\bar{\rho} = \int_{\mathbb{R}^3 \times \mathbb{R}_+} (q_e f_e I^{\alpha_e} + q_i f_i I^{\alpha_i}) dv dI = n_e q_e + n_i q_i, \quad (19)$$

$$j = \int_{\mathbb{R}^3 \times \mathbb{R}_+} v (q_e f_e I^{\alpha_e} + q_i f_i I^{\alpha_i}) dv dI = n_e q_e u_e + n_i q_i u_i. \quad (20)$$

2.2.3. *Properties of the model.*

Proposition 1. *The model (12, 13, 14, 15, 16) conserves the mass per species, the total impulsion and the total energy.*

The proof is straightforward and based as in [2] on the definition of the fictitious quantities (15, 16).

Theorem 2.1. *The model (12, 13, 14, 15, 16) satisfies a H theorem.*

The model satisfies an entropic inequality

$$\begin{aligned} & \frac{1}{\tau_e} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\mathcal{M}_e(f_e) - f_e) \ln(f_e) I^{\alpha_e} dv dI + \frac{1}{\tau_i} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\mathcal{M}_i(f_i) - f_i) \ln(f_i) I^{\alpha_i} dv dI \\ & + \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\overline{\mathcal{M}}_e(f_e, f_i) - f_e) \ln(f_e) I^{\alpha_e} dv dI + \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\overline{\mathcal{M}}_i(f_e, f_i) - f_i) \ln(f_i) I^{\alpha_i} dv dI \leq 0. \end{aligned}$$

The equality holds in the above equation if and only if there exists $(n_e, n_i, u, T) \in \mathbb{R}_+^2 \times \mathbb{R}^3 \times \mathbb{R}_+$ such that

$$\mathcal{M}_\beta = \frac{n_\beta}{(2\pi \frac{k_B}{m_\beta} T)^{\frac{3}{2}}} \frac{1}{\mathcal{Q}_\beta(T)} \exp\left(-\frac{(v-u)^2}{2k_B \frac{T}{m_\beta}} - \frac{I}{k_B T}\right), \quad \beta \in \{e, i\}.$$

An important feature of the polyatomic model (12, 13, 14, 15, 16) is that it satisfies an entropy dissipation property that is compatible with the macroscopic ones. The entropy dissipation property has already been obtained in ([2]) for the system (1), starting directly from the fluid system. In the present case, we are able to show that the entropy of the system (1) is compatible with the Boltzmann entropy.

3. Construction of the fluid model.

3.1. Scaling on the one dimensional BGK model. Suppose that the system is space homogeneous in two directions. Hence, the distribution function f_β of species β depends on time $t \in \mathbb{R}_+$, space $x \in \mathbb{R}$, velocity $v_1 \in \mathbb{R}$ and on the energy variable $I \in \mathbb{R}_+$. The model (12, 13, 14, 15, 16) can be rewritten in this case

$$\begin{cases} \partial_t f_\beta + v_1 \partial_x f_\beta + \frac{q_\beta}{m_\beta} E \partial_{v_1} f_\beta &= \frac{1}{\varepsilon} (\mathcal{M}_\beta - f_\beta) + \frac{1}{\tau_{\beta, \delta}} (\overline{\mathcal{M}}_\beta - f_\beta), \beta \neq \delta \\ \partial_t E &= -\frac{j}{\varepsilon^2}, \\ \partial_x E &= \frac{p}{\varepsilon^2}, \end{cases} \quad (21)$$

where ε is a nonnegative parameter proportional to the Knudsen number. In that case, Maxwellian distributions (13, 14) become

$$\mathcal{M}_\beta = \frac{n_\beta}{(2\pi \frac{k_B}{m_\beta} T_\beta)^{\frac{3}{2}}} \frac{1}{\mathcal{Q}_\beta(T_\beta)} \exp\left(-\frac{(v_1 - u_\beta)^2 + v_2^2 + v_3^2}{2k_B \frac{T_\beta}{m_\beta}} - \frac{I}{k_B T_\beta}\right), \quad (22)$$

$$\overline{\mathcal{M}}_\beta(f_e, f_i) = \frac{n_\beta}{(2\pi \frac{k_B}{m_\beta} T^\#)^{\frac{3}{2}}} \frac{1}{\mathcal{Q}_\beta(T^\#)} \exp\left(-\frac{(v_1 - u_\beta^\#)^2 + v_2^2 + v_3^2}{2k_B \frac{T^\#}{m_\beta}} - \frac{I}{k_B T^\#}\right) \quad (23)$$

where $u^\#$ and $T^\#$ are defined in (15) and (16).

3.2. Hydrodynamic limit.

Proposition 2. *The system (21, 22, 23, 15, 16) converges formally to the non conservative bitemperature Euler system where E is given by Ohm's law*

$$\frac{1}{\rho_e} \partial_x p_e - \frac{1}{\rho_i} \partial_x p_i = \left(\frac{n_e q_e}{\rho_e} - \frac{n_i q_i}{\rho_i}\right) E = \frac{\rho}{\rho_e \rho_i} n_e q_e E = -\frac{\rho}{\rho_e \rho_i} n_i q_i E \quad (24)$$

and

$$\nu_{ei} = \frac{k_B (\frac{5}{2} + \alpha_e) (\frac{5}{2} + \alpha_i) n_e n_i}{(\frac{5}{2} + \alpha_e) n_e + (\frac{5}{2} + \alpha_i) n_i} \frac{1}{\tau_{ei}}. \quad (25)$$

Performing a Chapman-Enskog expansion, it comes that

$$f_\beta = \mathcal{M}_\beta + \varepsilon g_\beta, \quad (26)$$

with the constraints

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} g_\beta I^{\alpha_\beta} dv dI = 0, \quad \int_{\mathbb{R}^3 \times \mathbb{R}_+} v_1 (m_i g_i I^{\alpha_i} + m_e g_e I^{\alpha_e}) dv dI = 0, \quad (27)$$

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} \left(\frac{1}{2} m_\beta v^2 + I \right) I^{\alpha_\beta} dv dI = 0. \quad (28)$$

Moreover, Gauss equation (21) implies that $n_i q_i = n_e q_e + \mathcal{O}(\varepsilon^2)$. So, $n_i = Z n_e + \mathcal{O}(\varepsilon^2)$. Plugging (26) into equation (12) leads to

$$\partial_t \mathcal{M}_\beta + v_1 \partial_x \mathcal{M}_\beta + \frac{q_\beta}{m_\beta} E \partial_{v_1} \mathcal{M}_\beta = -g_\beta + \frac{1}{\tau_{ei}} (\overline{\mathcal{M}}_\beta - \mathcal{M}_\beta) + \mathcal{O}(\varepsilon), \quad \beta \in \{e; i\}.$$

Mass conservation equation is obtained by integrating w.r.t v and I . Next, Ampère equation leads to

$$\begin{aligned} \partial_t j &= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (q_e m_e v_1 \mathcal{M}_e I^{\alpha_e} + q_i m_i v_1 \mathcal{M}_i I^{\alpha_i}) dv dI \\ &+ \frac{1}{\varepsilon} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (q_e m_e v_1 g_e I^{\alpha_e} + q_i m_i v_1 g_i I^{\alpha_i}) dv dI + \mathcal{O}(1). \end{aligned}$$

The right-hand side can be rewritten

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} (q_e g_e v_1 \mathcal{M}_e I^{\alpha_e} + q_i g_i v_1 \mathcal{M}_i I^{\alpha_i}) dv dI = \mathcal{O}(\varepsilon).$$

So, combining the previous equation with (27), we get

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} q_\beta g_\beta v_1 \mathcal{M}_\beta I^{\alpha_\beta} dv dI = \mathcal{O}(\varepsilon), \quad \beta \in \{e; i\}.$$

The equation of the conservation of the impulsions is recovered. Moreover, proceeding as in ([2]), we get Ohm's law (24). Next the energy equation on the electrons writes

$$\begin{aligned} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\partial_t \mathcal{M}_e + v_1 \partial_x \mathcal{M}_e) \left(\frac{1}{2} m_e v^2 + I \right) I^{\alpha_e} dv dI &+ \int_{\mathbb{R}^3 \times \mathbb{R}_+} \frac{q_e}{m_\beta} E \partial_{v_1} \mathcal{M}_e \left(\frac{1}{2} m_e v^2 + I \right) I^{\alpha_e} dv dI \\ &= \frac{1}{\tau_{ei}} \int_{\mathbb{R}^3 \times \mathbb{R}_+} (\overline{\mathcal{M}}_e - \mathcal{M}_e) \left(\frac{1}{2} m_e v^2 + I \right) I^{\alpha_e} dv dI. \end{aligned}$$

Moreover

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} (\overline{\mathcal{M}}_e - \mathcal{M}_e) \left(\frac{1}{2} m_e v^2 + I \right) I^{\alpha_e} dv dI = \left(\frac{5}{2} + \alpha_e \right) n_e k_B (T - T_e).$$

So, according to the relation (9) defining T , we get

$$\int_{\mathbb{R}^3 \times \mathbb{R}_+} (\overline{\mathcal{M}}_e - \mathcal{M}_e) \left(\frac{1}{2} m_e v^2 + I \right) I^{\alpha_e} dv dI = \frac{\left(\frac{5}{2} + \alpha_e \right) n_e \left(\frac{5}{2} + \alpha_i \right) n_i}{\left(\frac{5}{2} + \alpha_e \right) n_e + \left(\frac{5}{2} + \alpha_i \right) n_i} k_B (T_i - T_e)$$

and ν_{ei} is given by (25).

3.3. Entropy dissipation. As proved in [2], the system (1) owns a dissipative entropy-entropy flux pair

$$\eta = \eta_e + \eta_i, \quad Q = u \eta \quad (29)$$

where

$$\eta = - \frac{\rho_\alpha}{m_\alpha(\gamma_\alpha - 1)} \left[\ln \left(\frac{(\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha}{\rho_\alpha^{\gamma_\alpha}} \right) + C \right]. \quad (30)$$

Here, C is a nonnegative constant. With the same method as in [2] we can prove

Theorem 3.1. *Let (η, Q) be defined by (29)-(30). η is a strictly convex dissipative entropy for system (1) and Q is the related entropy flux. More precisely, any smooth solution of the system satisfies the following equality:*

$$\partial_t \eta(\mathcal{U}) + \partial_x Q(\mathcal{U}) = - \frac{\nu_{ei}}{k_B T_i T_e} (T_i - T_e)^2. \quad (31)$$

4. Some exact solutions. In order to test the numerical approximation of the left hand side of (1) we compute contact discontinuities and rarefaction waves for the related homogeneous system.

4.1. Contact discontinuities. A contact discontinuity is a weak solution $\mathcal{U} = (\rho, \rho u, \mathcal{E}_e, \mathcal{E}_i)$ of (32) such that u is a constant and

$$\rho(x, t) = \begin{cases} \rho_L & \text{if } x < ut, \\ \rho_R & \text{if } x > ut \end{cases} \quad \mathcal{E}_\beta(x, t) = \begin{cases} \mathcal{E}_{\beta,L} & \text{if } x < ut, \\ \mathcal{E}_{\beta,R} & \text{if } x > ut \end{cases} \quad \beta = e, i.$$

Here, $\rho_L, \rho_R, \mathcal{E}_{\beta,L}, \mathcal{E}_{\beta,R}$ are constant. In that case the homogeneous system related to (1) can be written under the conservative form:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p_e + p_i) = 0, \\ \partial_t \mathcal{E}_e + \partial_x(u(\mathcal{E}_e + c_e(p_e + p_i))) = 0, \\ \partial_t \mathcal{E}_i + \partial_x(u(\mathcal{E}_i + c_i(p_e + p_i))) = 0. \end{cases} \quad (32)$$

Rankine-Hugoniot jump conditions are

$$[u] = 0, \quad [p_e + p_i] = 0,$$

One can realize those conditions by taking $[\rho] = 0$ or not. In the first case, nontrivial contact discontinuities are obtained with nonequal left and right values of the ionic and electronic pressures. This case is specific to the bi-temperature model. The case $[\rho] \neq 0$ appears also in contact discontinuities for the 3×3 mono-temperature Euler system.

For contact discontinuities there is no entropy dissipation:

$$\partial_t \eta(\mathcal{U}) + \partial_x Q(\mathcal{U}) = 0.$$

4.2. Rarefaction waves. A rarefaction wave is a selfsimilar continuous, piecewise C^1 solution of (1) with $\nu_{ei} = 0$. As we look for a smooth solution, we may use the variable $\mathcal{V} = (\rho, u, \varepsilon_e, \varepsilon_i)$. The rarefaction waves are given by solutions $\mathcal{V}(x, t) = \mathcal{W}\left(\frac{x}{t}\right)$ of the homogeneous system related to (4) that is, denoting $y = x/t$:

$$\{-y I + A(\mathcal{W}(y))\} \mathcal{W}'(y) = 0 \quad (33)$$

where A is given by (5). Rarefaction waves are closely related to the integral curves of the eigenvectors of A , as soon as the fields are genuinely nonlinear, see [6] for

example. Let us consider the eigenvalue $\lambda_+ = u + a$ with a given by (6), with the eigenvector r_4 satisfying (7). We solve the ODS

$$\Phi'(\xi) = r_4(\Phi(\xi)), \quad \Phi(0) = \mathcal{W}_L.$$

We set $\mathcal{W}_R = \Phi(\xi_0)$ with $\xi_0 > 0$ and $\Psi(\xi) = \lambda(\Phi(\xi))$. Ψ is an increasing monotone function. We set

$$\mathcal{W}(y) = \begin{cases} \mathcal{W}_L & \text{si } y \leq \lambda_+(\mathcal{W}_L), \\ \Phi(\Psi^{-1}(y)) & \text{if } \lambda_+(\mathcal{W}_L) \leq y \leq \lambda_+(\mathcal{W}_R), \\ \mathcal{W}_R & \text{si } y \geq \lambda_+(\mathcal{W}_R). \end{cases}$$

We have $\mathcal{W} \in C^1(\mathbb{R})$ and (33). Hence we have to solve for $\xi > 0$:

$$\begin{cases} \rho'(\xi) = \rho(\xi), \\ u'(\xi) = a(\xi), \\ \varepsilon'_\beta(\xi) = (\gamma_\beta - 1)\varepsilon_\beta(\xi), \quad \beta = e, i. \end{cases} \quad (\rho(0), u(0), \varepsilon_e(0), \varepsilon_i(0)) = \mathcal{W}_L.$$

We find

$$\begin{cases} \rho(\xi) = \rho_L e^\xi, \\ u'(\xi) = \sqrt{\sum_\beta \gamma_\beta (\gamma_\beta - 1) c_\beta \varepsilon_{\beta,L} e^{(\gamma_\beta - 1)\xi}}, \\ \varepsilon_\beta(\xi) = \varepsilon_{\beta,L} e^{(\gamma_\beta - 1)\xi}, \quad \beta = e, i. \end{cases} \quad (34)$$

As

$$T_\beta = \frac{(\gamma_\beta - 1)m_\beta}{k_B} \varepsilon_\beta \quad \text{and} \quad p_\beta = (\gamma_\beta - 1)\rho_\beta \varepsilon_\beta,$$

we have also:

$$T_\beta(\xi) = T_{\beta,L} e^{(\gamma_\beta - 1)\xi} \quad \text{et} \quad p_\beta(\xi) = p_{\beta,L} e^{\gamma_\beta \xi}, \quad \beta = e, i.$$

If $\gamma_i \neq \gamma_e$ we cannot parametrize the rarefaction curves by the pressure as one does for the monotemperature Euler system. Hence from (34), the rarefaction curves can be parametrized as follows

$$\xi = \ln\left(\frac{\rho}{\rho_L}\right), \quad \varepsilon_\beta = \varepsilon_{\beta,L} \left(\frac{\rho}{\rho_L}\right)^{\gamma_\beta - 1}.$$

So

$$T_\beta = T_{\beta,L} \left(\frac{\rho}{\rho_L}\right)^{\gamma_\beta - 1}.$$

We retrieve the fact that $\rho_R > \rho_L$. We have also $p_\beta > p_{\beta,L}$ and the specific entropy by species is constant. It is defined by

$$S_\beta = \frac{p_\beta}{\rho_\beta^{\frac{1}{\gamma_\beta}}}.$$

For all ξ :

$$S_\beta(\xi) = S_\beta(0), \quad \beta = e, i$$

hence

$$S_{\beta,L} = S_{\beta,R}, \quad \beta = e, i.$$

5. Numerical approximation. In this section we derive a numerical scheme starting from a semi-discretization of the kinetic model. In ([2]), this approach has been developed for a monoatomic gas mixture. Moreover by using the formalism of discrete BGK models ([3]), an analogous scheme can be obtained including the polyatomic case.

The spacial discretisation is defined by the step Δx and the cells $C_j =]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$. We consider that Δx is constant whereas the time step Δt : $t_0 = 0$, $t_{n+1} = t_n + \Delta t$ can be variable.

We use a finite volume approach: for any unknown $V(x, t)$, we look for the approximations V_j^n of the average of V at timet t_n on the cell C_j .

The initial condition \mathcal{U}^0 being given, for $\beta = e, i$ we set $\rho_\beta^0 = \rho^0 c_\beta$, $n_\beta^0 = \rho_\beta^0 / m_\beta$, and $u_\beta^0 = u^0$.

Suppose that \mathcal{U}^n is known. For $\beta = e, i$ we set $\overline{\rho_{\beta,j}^n} = c_\beta \rho_j^n$ and $\overline{U_{\beta,j}^n} = (\overline{\rho_{\beta,j}^n}, \overline{\rho_{\beta,j}^n} u_j^n, \mathcal{E}_{\beta,j})$.

So, $f_e^n(v, I)$, $f_i^n(v, I)$ can be computed according to

$$f_\beta^n(v, I) = \mathcal{M}_\beta(\overline{U_{\beta,j}^n}), \quad \beta = e, i. \quad (35)$$

For P_β defined by

$$P_\beta(f_\beta) = \int_{\mathbb{R}^3} \begin{pmatrix} m_\beta \\ m_\beta v \\ (m_\beta \frac{v^2}{2} + I) \end{pmatrix} I^{\alpha_\beta} f_\beta I^{\alpha_\beta} dv dI \quad (36)$$

one has

$$P_\beta(f_\beta^n) = \overline{U_{\beta,j}^n}, \quad P_\beta(v_1 \mathcal{M}_\beta(\overline{U_{\beta,j}^n})) = F_\beta(\overline{U_{\beta,j}^n}), \quad (37)$$

where F_β is the flux of 3×3 Euler equations with γ_β law.

Definition 5.1. For any value of v and I , $h_{\beta,j+\frac{1}{2}}(v, I) = h_\beta(f_{\beta,j}(v, I), f_{\beta,j+1}(v, I), v, I)$ is a numerical flux such that for any v, I , $h_\beta(\cdot, \cdot, v, I)$ is Lipschitz continuous, and for any f :

$$h_\beta(f, f, v, I) = v_1 f. \quad (38)$$

We then follow the lines of ([2]). Using the flux h_β we approximate the kinetic equation in (21) with respect to (x, t) , keeping continuous the dependance on (v, I) . We apply P_e on f_e , P_i on f_i , and obtain $U_{e,j}^{n+1}$ and $U_{i,j}^{n+1}$. The quasi-neutrality conditions allow us to show that $\rho_{\beta,j}^{n+1} = c_\beta \rho_j^{n+1}$ for $\beta = e, i$. Denoting for $\beta = e, i$

$$F_{\beta,j+\frac{1}{2}} = \mathcal{F}_\beta(U_{\beta,j}, U_{\beta,j+1}), \quad \mathcal{F}_\beta(U_\beta, V_\beta) = P_\beta(h_\beta(\mathcal{M}_\beta(U_\beta), \mathcal{M}_\beta(V_\beta), \cdot)), \quad (39)$$

we then obtain a discrete Ohm's law involving

$$\delta_{j+\frac{1}{2}}^n = -c_i F_{e,j+\frac{1}{2},2}^n + c_e F_{i,j+\frac{1}{2},2}^n. \quad (40)$$

Finally, defining $F_{j+\frac{1}{2}}$ by

$$F_{j+\frac{1}{2}} = \begin{pmatrix} F_{e,j+\frac{1}{2},1} + F_{i,j+\frac{1}{2},1} \\ F_{e,j+\frac{1}{2},2} + F_{i,j+\frac{1}{2},2} \\ F_{e,j+\frac{1}{2},3} \\ F_{i,j+\frac{1}{2},3} \end{pmatrix}. \quad (41)$$

We get a scheme that is consistent with the Euler system (1):

Proposition 3. For any $n \geq 0$ if $\mathcal{U}^n = \{\mathcal{U}_j^n\}_{j \in \mathbb{Z}}$ is the approached solution of the system (1) at time t_n , we set

$$U_{\beta,j}^n = (c_\beta \rho_j^n, c_\beta \rho_j^n u_j^n, \mathcal{E}_\beta), \quad \beta = e, i. \quad (42)$$

A kinetic flux h_β is chosen as in the definition 5.1. The numerical fluxes $F_{\beta,j+\frac{1}{2}}$, $F_{j+\frac{1}{2}}$ and $\delta_{j+\frac{1}{2}}$ are then defined by (39), (40), (41). The approach solution at time t_{n+1} is defined by the implicit scheme

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2},1}^n - F_{j-\frac{1}{2},1}^n), \\ \rho_j^{n+1} u_j^{n+1} = \rho_j^n u_j^n - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2},2}^n - F_{j-\frac{1}{2},2}^n), \\ \mathcal{E}_{e,j}^{n+1} = \mathcal{E}_{e,j}^n - \frac{\Delta t}{\Delta x} (F_{e,j+\frac{1}{2},3}^n - F_{e,j-\frac{1}{2},3}^n) - u_j^{n+1} \frac{\Delta t}{\Delta x} (\delta_{j+\frac{1}{2}}^n - \delta_{j-\frac{1}{2}}^n) \\ \quad + \Delta t \nu_{ei} (T_{i,j}^{n+1} - T_{e,j}^{n+1}), \\ \mathcal{E}_{i,j}^{n+1} = \mathcal{E}_{i,j}^n - \frac{\Delta t}{\Delta x} (F_{i,j+\frac{1}{2},3}^n - F_{i,j-\frac{1}{2},3}^n) + u_j^{n+1} \frac{\Delta t}{\Delta x} (\delta_{j+\frac{1}{2}}^n - \delta_{j-\frac{1}{2}}^n) \\ \quad - \Delta t \nu_{ei} (T_{i,j}^{n+1} - T_{e,j}^{n+1}). \end{cases} \quad (43)$$

In the present paper we specify the following kinetic flux

$$h_\beta(f, g, v, I) = v_1 \left(\frac{\lambda_3}{\lambda_3 - \lambda_1} f(v, I) - \frac{\lambda_1}{\lambda_3 - \lambda_1} g(v, I) \right) - \frac{\lambda_1 \lambda_3}{\lambda_3 - \lambda_1} (g(v, I) - f(v, I)), \quad (44)$$

where λ_1 and λ_3 are constants to be fixed. This flux gives rise to a HLL scheme that is computed in the following subsection.

6. Numerical results. In the following test cases, the kinetic HLL scheme corresponding to the kinetic flux (44) is implemented and tested.

6.1. Implosion test case. Consider a Riemann problem with

$$\begin{aligned} \rho_- = 1, \quad \rho_+ = 1, \quad u_- = 0, \quad u_+ = 0, \\ T_{e,-} = 2.3 \times 10^6, \quad T_{e,+} = 2.3 \times 10^7, \quad T_{i,-} = 1.7406 \times 10^6, \quad T_{i,+} = 1.7406 \times 10^7. \end{aligned}$$

We set $\gamma_e = \gamma_i = \frac{5}{3}$, $Z = 1$ and a final computation time $t = 4.0901 \times 10^{-7}$ sec. The coefficient ν_{ei} is given in the NRL plasma formulary [15], p.34. This case has been studied in [11]. The final time is such that the asymptotic value $T_e - T_i = 0$ is numerically reached. As $\gamma_e = \gamma_i$, (ρ, u, p) is known analytically as solution of the monotemperature Euler system. The results are depicted in figure 1 for density and velocity, in figure 2 for the total pressure $p = p_e + p_i$ and electronic and ionic temperatures. (ρ, u, p) is very close to its exact value. The temperatures take the same values as in [11].

Then we take the same Riemann data but we set $\gamma_e = 5/3$, $\gamma_i = 7/5$, the other parameters being unchanged. Here we have no analytical solution to compare with. (ρ, u, p) changes with the value of ν_{ei} , while in the monoatomic case it does not. In figure 3 we show the values of ρ and u for $\nu_{ei} = 0$ and for ν_{ei} as in the first test case. One can observe that the propagation speed of the waves is slightly modified. In figure 4 the electronic and ionic temperatures are depicted for $\nu_{ei} = 0$ and $\nu_{ei} \neq 0$. In this last case, we observe again that the limit value $T_e - T_i = 0$ is reached.

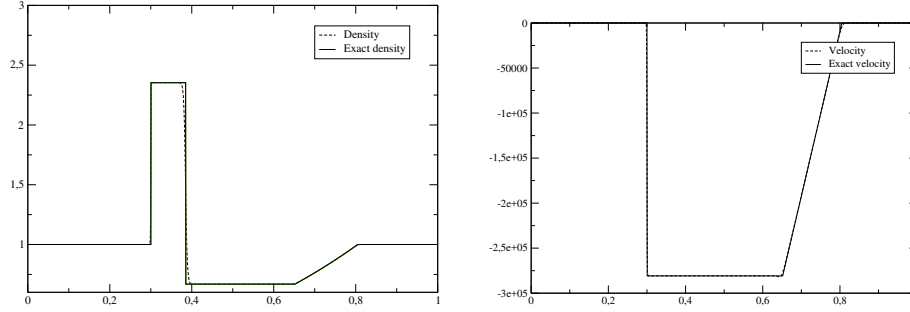


FIGURE 1. Implosion with $\gamma_e = \gamma_i$. Comparison with exact solutions Left: density. Right: velocity.

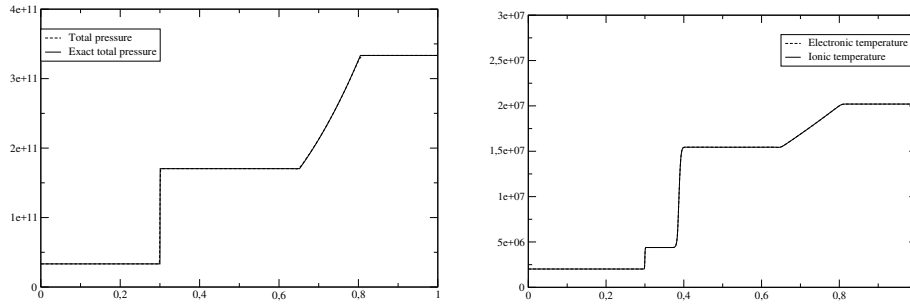


FIGURE 2. Implosion with $\gamma_e = \gamma_i$. Left: exact and computed total pressures. Right: electronic and ionic temperatures.

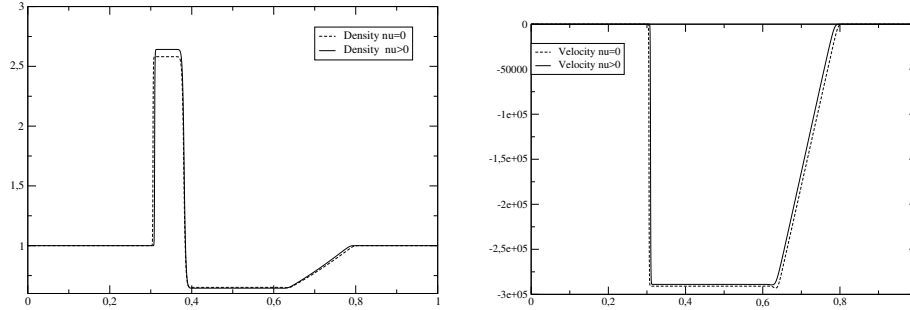


FIGURE 3. Implosion with $\gamma_e \neq \gamma_i$. Comparison between $\nu_{ei} = 0$ and $\nu_{ei} \neq 0$. Left: density. Right: velocity.

6.2. Double rarefaction. We stay with data of same magnitude as for the implosion case:

$$\begin{aligned} \rho_- &= 1, & \rho_+ &= \rho_-, & u_- &= -30000, & u_+ &= -u_-, \\ T_{e,-} &= 2.3 \times 10^7 & T_{e,+} &= T_{e,-}, & T_{i,-} &= 1.7406 \times 10^7, & T_{i,+} &= T_{i,-}. \end{aligned}$$

We set $\gamma_e = 5/3$ and $\gamma_i = 7/5$, $Z = 1$ and a final computation time $t = 4.0901 \times 10^{-7} \text{ sec}$. We set $\nu_{ei} = 0$ so that the solution consists of two rarefaction waves propagating to left and right, the contact discontinuity being trivial. In order to determine the analytical solution, denoting $(\rho, u, \varepsilon_e, \varepsilon_i)$ the intermediate state, one

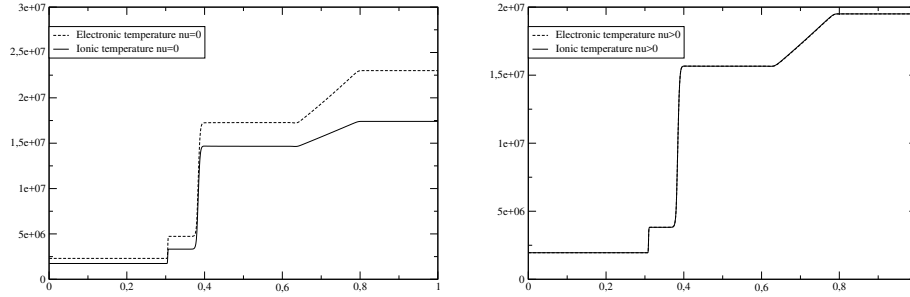


FIGURE 4. Implosion with $\gamma_e \neq \gamma_i$, electronic and ionic temperatures. Left: $\nu_{ei} = 0$. Right: $\nu_{ei} \neq 0$.

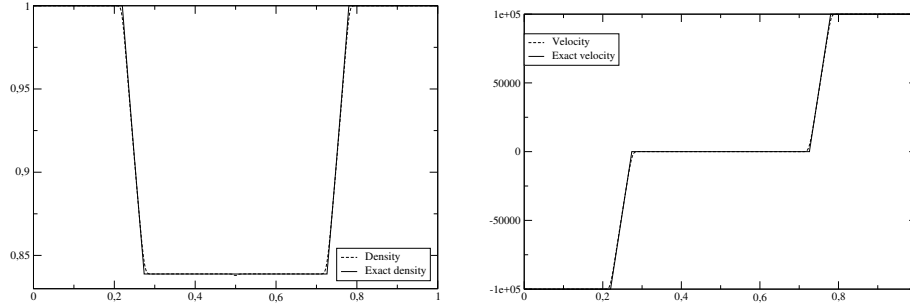


FIGURE 5. Double rarefaction. Left: density. Right: velocity.

has to find $\xi > 0$ such that:

$$\begin{cases} \rho = \rho_{\pm} e^{-\xi} \\ u = u_{-} + \int_0^{\xi} a(s) ds = u_{+} - \int_0^{\xi} a(s) ds \\ \varepsilon_{\beta} = e^{-(\gamma_{\beta}-1)\xi} \varepsilon_{\beta, \pm}, \quad \beta = e, i, \end{cases}$$

with

$$a(s) = \left(\sum_{\beta} \gamma_{\beta} (\gamma_{\beta} - 1) c_{\beta} \varepsilon_{\beta, \pm} e^{-(\gamma_{\beta}-1)s} \right)^{1/2}.$$

Hence we find numerically $\xi > 0$ such that

$$u_{+} = \int_0^{\xi} a(s) ds.$$

The numerical results are depicted in figures 5 and 6. We compare the exact and computed results for density, velocity and temperatures. As already observed in [2] for $\gamma_e = \gamma_i = 5/3$, a peak of ionic temperature happens at $x = 1/2$.

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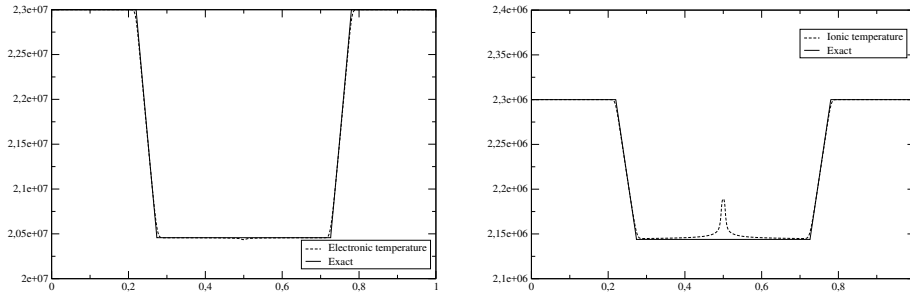


FIGURE 6. Double rarefaction. Left: electronic temperature.
Right : ionic temperature.

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